1970

A laboratory method to teach geometry in selected sixth grade mathematics classes

Jack Dale Wilkinson
Iowa State University

71-7342

WILKINSON, Jack Dale, 1931-
A LABORATORY METHOD TO TEACH GEOMETRY IN SELECTED SIXTH GRADE MATHEMATICS CLASSES.

Iowa State University, Ph.D., 1970
Education, theory and practice

University Microfilms, Inc., Ann Arbor, Michigan

A LABORATORY METHOD TO TEACH GEOMETRY IN SELECTED SIXTH GRADE MATHEMATICS CLASSES

by

Jack Dale Wilkinson

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Education

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Head of Major Area

Signature was redacted for privacy.

Dean of Graduate College

Iowa State University
Ames, Iowa

1970

TABLE OF CONTENTS

INTRODUCTION

The past fifteen years has witnessed wide-spread activity relative to new mathematics curricula for elementary, junior high, and high school pupils.

E. G. Begle (6, p. 1), director of the School Mathematics Study Group, in summarizing the revolution in school mathematics states:

> New school mathematics differ very little from old ones as far as subject matter is concerned. Only a few old topics have been de-emphasized and only a few new topics have been added. The chief difference between the old and the new is the point of view toward mathematics. Now there is an equal emphasis on an understanding of the basic concepts of mathematics and their interrelationships, i.e., the structure of mathematics.

The post-Sputnik drive was clearly one to improve the quality of content in the mathematics curriculum. By contrast, however, there has been much less activity to improve the methods of teaching mathematics.

Need for the Study

What has commonly been called the "revolution in school mathematics" essentially ended about 1965. More recently, mathematics educators in colleges and universities, along with classroom teachers, have become concerned with the methods used to teach the new content which evolved from the "revolution."

Scott (56, p. 15) has summarized the basic principles of the new pedagogy for teaching mathematics in the following ten statements:

1. The structure of mathematics should be stressed at all levels.

2. Children are capable of learning more abstract and more complex concepts when the relationships between concepts are stressed.

3. Existing arithmetic programs may be severely condensed because children are capable of learning concepts at much earlier ages than formerly thought.

4. Any concept may be taught a child of any age in some intellectually honest manner, if one is able to find the proper language for expressing the concept.

5. The inductive approach or the discovery method is logically productive and should enhance learning and retention.

6. The major objective of a program is the development of independent and creative thinking processes.

7. Human learning seems to pass through the stages of preoperations, concrete operations, and formal operations.

8. Growth of understanding is dependent upon concept exploration through challenging apparatus and concrete materials and cannot be restricted to mere symbolic manipulations.

9. Teaching mathematical skills is regarded as a tidying-up of concepts developed through discovery rather than by a step by step process for memorization.

10. Practical application of isolated concepts or systems of concepts, particularly those drawn from the natural sciences, are valuable to reinforcement and retention.

Since 1965, discovery teaching, guided discovery, programmed instruction, computer assisted instruction, and inquiry training have been the subject of much study and research. Still more recently, individually guided instruction and activity programs in mathematics have been the subject of action research in many classrooms. A quick glance through professional journals in the field of mathematics education will illustrate the activity and interest in laboratory-type programs.

Studies in discovery teaching and learning exhibit a trend that favors a sequence of learning from examples, followed by the conceptualization of a formal rule. These studies have dealt with a traditional classroom setting, one where the teacher states the rule and examples as a control, while the experimental group has more class participation in moving from an example to a rule. In both cases the materials in the classrooms have been of the pencil-paper, chalk-blackboard variety.

Research is needed in analyzing the role which activity plays in the discovery process. Commercial textbook companies are producing laboratory-type units for elementary and junior high school mathematics classes. The use of laboratory units is currently receiving a great deal of attention from mathematics teachers. In light of the long-time interest in discovery teaching and learning, plus the newer dimension of activity learning through laboratory units, the need for studies to examine the effectiveness of such units seems imperative.

Purpose of the Study

The improvement of instruction in mathematics is of general concern to the writer. This study examined the relative effectiveness of a laboratory method of teaching geometry in selected sixth grade mathematics classes.

In any teaching situation, judgments must be made regarding the best methods, materials, and content organization to be presented to a given class by a given teacher. If teachers are presented with better materials and more options on methods, better teaching should be the net result.

Statement of the Problem

The problem of this investigation was to determine if the use of laboratory materials in teaching geometry to sixth grade pupils at Harlan Elementary School could be established to be significantly more effective than a more traditional method of instruction. The criterion variables were attitude toward mathematics, geometry achievement, and non-verbal intelligence.

More specifically, this study was to answer the following questions:

1. Can laboratory units be developed which teach the basic geometric content of a sixth grade mathematics program?
2. Can a laboratory method work effectively and efficiently in a sixth grade classroom?

3. Can teachers effectively use a laboratory method without the benefit of specific, formal preservice or inservice training?
4. Can pupils effectively make the transition to an activity-type mathematics curriculum using laboratory units?
5. Can pupils effectively use cassette tapes and cassette players to obtain directions and information to complete laboratory units?

This writer was interested in examining the effectiveness of three different methods of teaching geometry in selected sixth grade mathematics classes. In addition, however, the interaction of treatments with sex of pupils and I. Q. level of pupils was of interest.

In order to be able to examine main effects and pertinent interaction effects it was necessary to analyze the data by an analysis of covariance design which defines the pupil as the experimental unit.

The following set of twelve null hypotheses was tested under the above assumptions.

Null hypothesis 1   There will be no significant difference in pupil attitude toward mathematics due to treatments when initial differences between pupils have been adjusted with respect to attitude toward mathematics (attitude pre-test.)

Null hypothesis 2 There will be no significant difference in pupil attitude toward mathematics due to the interaction of treatments and sex of pupils when initial differences between pupils have been adjusted with respect to attitude toward mathematics.

Null hypothesis 3 There will be no significant difference in pupil attitude toward mathematics due to the interaction of treatments and I. Q. level of pupils when initial differences between pupils have been adjusted with respect to attitude toward mathematics.

Null hypothesis 4 There will be no significant difference in pupil attitude toward mathematics due to the second order interaction of treatments, sex of pupils, and I. Q. level of pupils when initial differences between pupils have been adjusted with respect to attitude toward mathematics.

Null hypotheses 5, 6, 7, and 8 are similar to the first four. The criterion variable for this set of hypotheses is geometry achievement.

Null hypothesis 5 There will be no significant difference in the geometry achievement of pupils due to treatments when initial differences between pupils have been adjusted with respect to geometry achievement (geometry achievement pre-test).

Null hypothesis 6 There will be no significant difference in the geometry achievement due to the interaction of treatments and sex of pupils when initial differences between pupils have been adjusted with respect to geometry achievement.

<u>Null hypothesis 7</u> There will be no significant difference in the geometry achievement of pupils due to the interaction of treatments and I. Q. level of pupils when initial differences between pupils have been adjusted with respect to geometry achievement.

<u>Null hypothesis 8</u> There will be no significant difference in the geometry achievement of pupils due to the second order interaction of treatments, sex of pupils, and I. Q. level of pupils when initial differences between pupils have been adjusted with respect to geometry achievement.

Null hypotheses 9, 10, 11, and 12 are similar to the above. The criterion variable in this set of hypotheses is the Lorge-Thorndike Intelligence Test.

<u>Null hypothesis 9</u> There will be no significant difference in the non-verbal intelligence test score of pupils due to treatments when initial differences between pupils have been adjusted with respect to non-verbal intelligence (non-verbal intelligence pre-test).

<u>Null hypothesis 10</u> There will be no significant difference in the non-verbal intelligence test score of pupils due to the interaction of treatments and sex of pupils when initial differences between pupils have been adjusted with respect to non-verbal intelligence.

Null hypothesis 11   There will be no significant difference in the non-verbal intelligence test score of pupils due to the interaction of treatments and I. Q. level of pupils when initial differences between pupils have been adjusted with respect to non-verbal intelligence.

Null hypothesis 12   There will be no significant difference in the non-verbal intelligence test score of pupils due to the second order interaction of treatments, sex of pupils, and I. Q. level of pupils when initial differences between pupils have been adjusted with respect to non-verbal intelligence.

The preceding twelve hypotheses were tested under the assumption that the pupil was the experimental unit ($n = 232$), i.e. an observation is a pupil's score on a test.

In planning this study, it was not possible to randomly assign pupils to classes. The pupils were assigned to classes by the elementary principal. He attempted to make the classes as heterogeneous as possible, but it was not a random assignment of pupils. Randomization in this study occurred when treatments were randomly assigned to classes.

With the class defined to be the experimental unit, the statistical model had a total of nine observations for each criterion variable. Each observation was a class mean on a test.

The following null hypotheses were tested using the class as the experimental unit.

<u>Null hypothesis 13</u> There will be no significant difference in attitude toward mathematics due to treatments when initial differences between class means have been adjusted with respect to attitude toward mathematics (attitude pre-test).

<u>Null hypothesis 14</u> There will be no significant difference in geometry achievement due to treatments when initial differences between class means have been adjusted with respect to geometry achievement (geometry achievement pre-test).

<u>Null hypothesis 15</u> There will be no significant difference in non-verbal intelligence test scores due to treatments when initial differences between class means have been adjusted with respect to non-verbal intelligence (non-verbal intelligence pre-test).

Definition of terms

In order to clarify the meanings of various terms used in this study, the following definitions are made.

<u>Control method</u> A teaching method which does not use laboratory units. This teaching method uses the teacher and the textbook as the basic sources of content and instruction.

<u>Laboratory method</u> A teaching method which allowed the pupil to work with manipulative materials. As far as possible, the pupil has an active role to play. The usual pattern is

to have pupils manipulate physical objects, then describe a pattern or rule based on an inductive sequence.

Laboratory-cassette method A teaching method which has the same basic format as the laboratory method. In addition to the work with manipulative materials, this method includes the use of cassette tapes and cassette players. All written directions and questions in a laboratory unit are recorded on a cassette tape. These pupils may read and/or listen to the directions and questions.

Pupils The subjects for this study met the following criteria:

a. Sixth grade pupil at Harlan Elementary School, Ames, Iowa, for the period of the study.

b. Completed all pre-tests and post-tests.

Laboratory unit A set of twenty-six different activity lessons in geometry. Each unit is housed in a separate box. This box contains all of the worksheets and manipulative materials needed for each learning experience.

Shoe box Another term for laboratory unit.

Activity learning School settings in which the learner develops mathematical concepts through active participation. This process may involve the manipulation of physical materials, the use of games, or experimenting with physical materials.

Discovery learning A form of learning whereby the pupil is actively involved in the process of formulating mathematical ideas. This is essentially a verbal process and does not necessarily require physical materials.

I. Q. level Each pupil was assigned to a level on the basis of his score on the Lorge-Thorndike Intelligence Test.

a. High I. Q. level: I. Q. scores in the range 117 to 142. There were 62 pupils in this level.

b. Middle I. Q. level: I. Q. scores in the range 105 to 116. There were 110 pupils in this level.

c. Low I. Q. level: I. Q. scores in the range from 79 to 104. There were 60 pupils in this level.

Delimitations of the Study

The scope of this investigation was confined to sixth grade pupils at Harlan Elementary School in Ames, Iowa, during the period from February 23, 1970, through March 20, 1970. There were nine sixth grade sections taught by three teachers. Each teacher taught a control group and two experimental groups. There were eighteen days of instruction and two days of testing.

Pupils were assigned to class sections on the basis of criteria formulated by the principal. In essence, these criteria were to make the classes as heterogeneous as possible.

The sixth grade program was semi-departmentalized with four blocks of time comprising the school day. Each of the four blocks was 90 minutes in length. One block was allotted f r science and mathematics; a second 90 minute block was assigned to social studies. Reading and language arts comprise block number three with the final 90 minute block assigned to special areas.

The schedule at Harlan Elementary School provided for approximately 50 minutes of mathematics instruction per day.

The total sixth grade enrollment was 251. The number of pupils which completed all pre-tests and post-tests was 232.

Organization of the Study

The material for this study has been divided into five chapters. The first chapter includes a background and setting for the study. The second chapter includes a summarization and analysis of related literature and research. Chapter three discusses the methodology and procedures for the study. The findings of the data collected in the study are examined in chapter four. The final chapter presents a summary, conclusions, and recommendations for further research.

REVIEW OF LITERATURE

This chapter cites literature and research pertinent to the problem being investigated, laboratory methods of teaching mathematics. In reviewing the literature and research relevant to this study, three general categories were examined: (1) a review of literature dealing with activity methods of teaching mathematics, (2) a review of research dealing with discovery teaching and discovery learning, and (3) a review of research dealing with laboratory methods used to teach mathematics.

Review of Activity Methods of Teaching Mathematics

In searching for better ways of teaching mathematics, this writer became interested in the writings of Jean Piaget, the noted Swiss psychologist.

Piaget (52) defines three periods in the development of intelligence. The sensory-motor period contains six stages and extends from birth to approximately age two years. The second period is the concrete operations period and contains two sub-periods. The first sub-period is the preoperational sub-period and extends from approximately two years of age through seven years of age. The second sub-period is the concrete operations sub-period and extends from approximately age seven through age eleven. The last of the three periods is called the formal operations period. This period lasts from approximately age eleven through age fifteen years.

Period one is concerned primarily with sensory-motor development, imitation, and play activities. The next period, concrete operations, brings about an essential difference in the child. In the sensory motor period the child is "relatively restricted to direct interactions with the environment," whereas in the later concrete operations period, the child is capable of "manipulating symbols that represent the environment" (51, p. 54).

The formal operations period, age eleven to fifteen years, begins where the concrete operations child left off--with concrete operations. The concrete operations child always starts with experience and makes limited interpolations and extrapolations from the data available to his senses. The adolescent, however, begins with the possible and then checks various possibilities against "memorial representations of past experiences," and eventually against sensory feedback from the concrete manipulations that are suggested by his hypotheses (51, p. 103).

The above theory proposes that pupils in the concrete operational stage operate on physical entities. The pupils would be called upon to make summaries and inductive judgments regarding the physical materials present. Next the pupils would anticipate what would happen in a hypothetical physical setting. This procedure would continue until the pupils were operating in the cognitive manner while completely divorced from any physical reinforcement of the cognition.

The Nuffield Mathematics Project (49, p. 113) states,

> The work of Piaget would seem to indicate that a majority of the children in primary schools are passing through what Piaget terms the stage of concrete operations, that they are able to deal confidently with the real problems arising from the use of concrete materials. This evidence produced by his team of research workers fully substantiates and justifies the belief that children learn through activity and experience.

Biggs (7, p. 9) states,

> Piaget emphasized two things about activity learning. First a child must be allowed to do things over and over again and thus reassure himself that what he has learned is true. Secondly, this practice should be enjoyable.

The Nuffield Project (28) has taken the findings of Piaget and incorporated them into a mathematics program for the elementary school.

The stress in the Nuffield Mathematics Project is on how to learn, not what to teach. Running through all the work is a central notion that children must be set free to make their own discoveries and think for themselves, and so achieve an understanding.

The Nuffield Project makes the point that if children are to achieve understanding they cannot go straight to abstractions. They must handle things.

The Madison Project (16) places heavy reliance on group discussion by children where the teacher serves essentially in the role as a moderator or discussion leader. There is extensive use of largely unstructured tasks such as "find the height of the school flag pole," in which the students are not told any method to use. It's up to them to devise a method. Children are also asked carefully devised sequences of questions which generally lead them to discover generalizations. The Madison Project leans heavily on getting children to learn from the structure of the subject itself.

In sharp contrast to the nondirective nature of Piaget and the Madison Project, Gagné and Ausubel support a theory of learning based on a well defined learning hierarchy.

Gagné's idea of learning hierarchy is important in analyzing a sequence of instructional moves. Gagné (24, p. 5) characterizes learning hierarchies as

> . . . an ordered set of intellectual skills such that each entity generates a substantial amount of positive transfer to the learning of a not previously acquired higher order capability.

Some of the important assumptions supporting Gagné's (25, p. 177) ideas about learning hierarchies are:

1. Any human task may be analyzed into a set of component tasks which are quite distinct from each other in terms of the experimental operations that are needed to perform them.

2. These task components are mediators of the final task performance; that is, their presence insures positive transfer into a final performance, and their absence reduces such transfer to near zero.

3. The basic principles of training design consist of:

a. identifying the component task of a final performance,

b. insuring that each of these component tasks is fully achieved, and

c. arranging the total learning situation in the sequence which will insure optimal mediational affects from one component to another.

Ausubel's position on learning hierarchies and sequencing is quite similar to Gagné's. Ausubel (2, p. 86) stated,

> Most complex tasks, particularly those that are sequential in nature, can be analyzed into a hierarchy of component learning sets or units . . . . This presupposes, of course, that the preceding step is always clear, stable, and well organized. If it is not the learning of all subsequent steps is jeopardized. Hence, new material in the sequences should never be introduced until all previous steps are thoroughly mastered.

A combination of the work of Piaget, the early writings of the Nuffield Project, and the Madison Project inspired this writer to produce some shoe boxes and to research these materials in a sixth grade mathematics setting.

Review of Research Dealing with Discovery Learning and Teaching

Mathematics educators have come to associate with Jerome Bruner such ideas as discovery, structure, and intuitive thinking. "No other single person has better embodied the letter and spirit of the psychology which undercurrents the new mathematics curricula" (6, p. 25).

For Bruner, the emphasis is on the kinds of <u>processes</u> learned by the student, in contrast to the specific subject matter <u>products</u> he may acquire. One quotation communicates the essence of the educational objectives for Bruner (9, p. 72).

> To instruct someone in these disciplines is not a matter of getting him to commit results to mind. Rather, it is to teach him to participate in the process that makes possible the establishment of knowledge. We teach a subject not to produce little living libraries on that subject, but rather to get a student to think mathematically for himself, to consider matters as a historian does, to take part in the process of knowledge getting.

Ausubel (3, p. 494), in reviewing the "more significant published research" dealing with discovery learning and teaching, listed the following conclusions:

1. The articles most commonly cited in the literature as reporting results supporting discovery techniques actually report no research findings whatsoever, and consist mainly of theoretical discussions, assertion, and conjecture or descriptions of existing programs utilizing discovery methods, and of enthusiastic but wholely subjective testimonials regarding the efficacy of discovery approaches.

2. Most of the reasonably well-controlled studies find neutral findings at best.

3. Most studies reporting positive findings either fail to control other significant variables or employ questionable techniques of statistical analysis.

Thus Ausubel argues, actual examination of the research literature allegedly supporting learning theory reveal that valid evidence to support discovery teaching is virtually nonexistent. Moreover, it appears that enthusiasts of discovery methods have been supporting each other by citing one another's opinions and assertions as evidence and by generalizing extravagantly from questionable findings.

Henderson (31, p. 1020), in discussing research on discovery teaching suggested, "One is tempted to admonish the reader to draw his own conclusions about the findings and conclusions of various experiments." Cronbach (13, p. 76) states,

> In spite of the confident endorsements of teachers of teaching through discovery that we read in semi-popular discourse on improving education, there is precious little substantive knowledge about what advantages it offers, and under what conditions the advantages accrue.

To some teachers, discovery is exemplified in some of Socrates' teachings. The teacher leads the pupil through a series of questions to which the pupil only needs respond "yes" or "no" in order to arrive at some understanding.

To other teachers, discovery means to put the students entirely on their own in seeking solutions to problems. The teacher provides no direction for the learning and the students must discover the solution.

A more moderate interpretation would be to have teachers interacting with students to seek solutions to problems. This method is often referred to as guided-discovery. The student does more than respond with a simple yes or no, yet he is not placed entirely "on his own" in seeking solutions to problems.

In three separate studies Scandura (54) examined the effects of discovery versus expository strategies for teaching. Two treatments were used to teach subjects to solve simple problems. The test consisted of novel problems (n) and routine problems (r). These tests were administered after each experiment. In the first study, using sixth grade pupils, the results of a t-test favored the mean performance of the discovery group on the n problems ($p < .01$). The discovery group received 153 minutes of instruction and the expository group had only 108 minutes. Scandura discusses the findings and attributes these differences to the combination of factors: (1) directiveness of presentation, (2) emphasis on meaning, (3) amount of problem solving practice, and (4) time at which the algorithm was introduced.

In the second study Scandura used fourth and fifth grade subjects. Efforts were made to make the expository teaching more meaningful and to make the presentation to the discovery group less direct. As before, the discovery class required more time (199 minutes as against 153). On this study, test results favored the expository group on the n-type problems.

In the third study, gifted fourth and fifth graders were subjects. Because of the small numbers of students and a non-normal distribution, the data was not analyzed. However, the raw scores favored the expository group.

Kersh (38) conducted an experiment which suggested that discovery teaching enhances motivation. Forty bright college students learned two rules of addition in elementary number theory. They learned these two rules of addition by three different methods: no help; directed reference (which used perceptual aids); and, rule given. Results of a chi-squared analysis after four weeks favored the no help group at the .05 level of significance. An analysis of the activities of the subjects during the four weeks following the experiment indicated that those subjects receiving no help practiced the rules more often than did the other groups.

In another study, Kersh (40) taught the distributive property for multiplication over addition to fifth graders. After sixteen training sessions, Kersh found no differences among subjects studying by free discovery, programmed discovery, and programmed guidance.

Fisher (21) describes three instructional methods employed in teaching elementary school mathematics to pupils in grades three, four, and five. The three instructional treatments were: individually prescribed instruction, programmed learning instruction, and standard classroom instruction. A major effort was made to present descriptive differences between the three curriculum treatments and to provide statistical data relevant to the arithmetic achievement of the pupils involved in those treatments.

Statistical data did not indicate any outstanding differences in achievement as evidenced by the pupils' scores on the standardized tests. However, the researcher and other participants in the project cited observational experience that the advantages of individually prescribed treatment outweighed the achievement test results, and that a continuing effort should be made to extend the values of individualizing instruction.

Moody (47) cites a study where he investigated the effectiveness of self-instructional reading materials, and in addition, compared student performance to teacher performance in a pre-test, treatment, post-test situation using content from non-metric geometry. Moody concludes that there is no support for the hypothesis that students who read materials in mathematics on their own will perform as well on selected tasks as those who have teachers explain and interpret content for them. There is support for the hypothesis that if a teacher performs at a certain level, that his students, following instruction, will perform at the same level on these tasks.

Wills (61) investigated the effect of learning by discovery on problem solving ability. Two weeks of instruction were presented to two different groups of intermediate algebra classes. Each day students in both groups were given rather difficult problems that required generalizations. One class had teacher guidance and discussions regarding methods used for discovering generalizations. In the other group, the instructor gave no such guidance. Students were pre-tested and post-tested on mathematical items not covered in the unit. Both groups did about equally well.

Worthen (64) designed a study to (a) identify and explore some of the teaching-learning variables that are operative in the discovery process, and (b) to compare a discovery and expository method in a classroom setting.

The sample consisted of fifth and sixth grade students. An extensive inservice program for teachers was used to teach procedures for using the instructional and evaluating materials in the study. Teachers were trained from two to six hours weekly for twenty weeks.

The expository method produced better initial learning ($p < .01$) than the discovery method. The discovery method produced better retention over five weeks ($p < .05$), and over eleven weeks ($p < .025$). No differences in transfer or in attitudes were present.

Stacey (58) studied the effects of directed versus independent discovery. He found that active participation and self-discovery were more productive in solving a group of simple problems which called for the sixth grade subjects to identify the one element in a set of five that did not belong. Craig (12), using college students, found results less favorable for the discovery method. His directed group, which received a brief introductory training period, learned better and retained better than the non-directed group. Kittell (42) conducted studies which yielded results similar to those of Craig's. These studies found that groups which had received an intermediate amount of directed learning experience were superior in learning retention and transfer to groups receiving either more or less directed learning.

Review of Research Dealing with Laboratory Methods Used to Teach Mathematics

The laboratory approach to teaching mathematics is by no means new. The progressive education movement, and more specifically John Dewey's work at the University of Chicago, incorporated real applied problems and the use of a laboratory to teach mathematics. In 1942, Morrison (48, p. 193) made the following statement regarding laboratory learning:

> Nearly, if not quite, every critical study of the utility of laboratories anywhere in the secondary school including junior college has shown that they have no utility over and above what can be achieved in lecture-table demonstrations.

In brief he concludes "laboratory work belongs to universities, and not in the field of general education."

The 22nd Yearbook of the National Council of Teachers of Mathematics, published in 1954, devoted a large part of that yearbook to laboratory teaching in mathematics. However, the writings dealt almost entirely with the use of audio-visual aids. Very little was said that would help a teacher develop the laboratory method as a strategy for teaching mathematics.

Wingo (63) pointed out that activity programs have often failed to produce any doing except following directions, and often have forgotten that one purpose of the activities is to provide the conditions for reflective thought, and further, that reflective thought itself is activity learning.

Davis (16, p. 21) believes that activity-type discovery material should accomplish the following goals:

1. Children should enjoy mathematics.
2. Children should have successful experiences with mathematics.
3. Children should approach mathematics creatively and not think in terms of following rote procedures.
4. Children should approach mathematics problems with determination, persistence, optimism, and confidence.

In analyzing the best approach to mathematics in the primary school the Nuffield Project asserts that we should follow the methods of the sciences (49). When a child first meets a new material in the sciences he experiments with it. This experimentation leads to some sort of informal hypothesis concerning the material. This empirical approach is the natural approach of a primary school child to his environment. It can be summarized as follows (49, p. 4): (a) free experimentation with material, (b) the formation of a hypothesis, (c) the testing of the hypothesis, and (d) the communication of findings.

The Nuffield Project, in talking about activity materials, states that they also present an exciting challenge to children who needed first to experiment quite freely with the new materials where no direction or even suggestions come from the teacher.

Holt (33) urges a liberal period of "messing around" in any science or activity oriented subject. It was generally suggested that the first step in a laboratory experience should be to provide the children with free play time.

Biggs and MacLean (7, p. 13) emphasize that "it's the attitude toward learning we are attempting to develop rather than the specific techniques such as workshop or laboratory methods." Later they state that

> Whether the type of classroom is described as laboratory, workshop, or activity approach does not matter. The important thing is an atmosphere which encourages resourcefulness, self-confidence, independence, patience, and competence. The children may be working individually or in groups. They will be doing different things--handling materials, measuring, discussing, and recording.

Dienes and Golding (20, p. 9) discuss the price we must pay to have "universal mathematical understanding." The authors state:

> The price is an abundance of materials. These materials are not intended for demonstration by the teacher, but as an essential tool in the learning armor of every child. There should be sufficient material in each classroom during the mathematics lesson for every child to have access to whatever he might need in trying to solve a problem. By encouraging children to work in groups, the cost of the equipment can be greatly reduced.

Hudgins (36) suggests that whether fifth grade mathematics students work in small groups or as individuals has no affect on their problem solving performance. Subjects in groups did solve more problems than those working independently, but when all students had to work independently there was no difference in performance.

May (46) in discussing learning laboratories in the elementary schools in Winnetka states,

> The main purpose of the learning laboratory is to help children become independent learners. Students are encouraged to look at patterns they have developed and then encouraged to predict results beyond the data they have acquired. There is no failure because all students are free to ask questions whenever they need help.

Davidson and Fare (15) describe the creation of a mathematics laboratory. In discussing the orientation of teachers to this method they make the following points: (1) that you can learn math not only with paper and pencil but also through the use of manipulative materials; (2) that the math lab approach involves active participation, exploration, hypothesizing, looking for patterns, and "doing" rather than being shown; (3) that mathematics is many things, that there are often many right answers to a problem, and that usually you can check your hunches yourself by means of the materials; (4) that although much of the work will seem like fun and games all of the lab experiences can be related to specific math concepts, to problem solving techniques, or to modes of mathematical thinking; (5) that at the beginning, the lab teacher will choose what activities you should embark on, but once you have pursued enough of the materials to know what some of the possibilities are, you will be given some choice;

(6) that care of the materials is the responsibility of each student; that the loss of one piece may mean that the entire set of materials is unusable; (7) that often projects will be started or materials introduced in the lab that will be followed up in a classroom or at home.

Clarkson (10, p. 494) supported the laboratory classroom environment because students could choose daily tasks from a wide variety of carefully planned situations. He cited three special advantages of the laboratories:

> 1. Piaget has made us aware of the developmental needs of children. While this subject is still quite controversial, wide agreement can probably be obtained from the thesis that children should have a very active experience with, say, measurement concepts before formal instruction begins.
>
> 2. Piaget emphasizes further, although his critics seem often to ignore this, that children develop, in their understanding of quantitative studies, very individually. And if development is highly individual, then this is one more reason why the laboratory situation, which provides an easy opportunity for students to choose tasks appropriate to their stage of development, is a good one.
>
> 3. The laboratory method allows children to communicate more easily and naturally with each other. Children are really great at explaining even highly complex sets of rules to each other.

Fitzgerald (23) describes a mathematics laboratory for prospective elementary school teachers. He credits much of the motivation for the laboratory at Michigan State to projects such as the Madison Project and the Nuffield Project. He cites three purposes of labs as they use them in the preparation of prospective elementary school teachers. The purposes are: (1) learning the mathematical concepts of the course, (2) becoming familiar with materials and how they are used, and (3) having a real experience in a student-centered rather than a teacher-centered classroom.

Beckland (5) investigated the effectiveness of an activity program in mathematics at grades four, five, and six. The experimental materials were prepared for investigation by pupils independent of teacher direction. At each grade level within each school, two classes studied these activity oriented materials: one used them independent of the teacher and the other studied them under teacher direction. A third class was given standard arithmetic materials and teaching techniques. The findings indicate that both methods of using the experimental materials provided the pupils with experience from which the pupils learned the ideas of this material. The pupils using the experimental materials were more able in adjusting to tasks requiring independent study skills than the pupils of the standard classes. In the comparison of these two methods, a meaningful expository approach to learning these experimental materials was at least as effective as the independent study of the materials.

Snyder (57) compared three methods of individualizing instruction in junior high mathematics. One program required the student to select the mathematical topics he would like to study during the year. A variety of materials were available and students were encouraged to select topics of value and interest to themselves. The second program required all students to participate. The student could choose from three different levels of assignments the level which seemed most appropriate for him. He could then supplement the work if he chose to. A third class was a conventional teacher taught class.

In the two experimental programs, the emphasis was on independent study and the teacher served as a resource person.

The control class had better gain scores on tests. However, both experimental groups scored better on reasoning tests than did the control classes.

Two studies dealt with actual activity learning by children as compared with the vicarious experience of watching the teacher demonstrate the activity. Toneym (59) studied fourth graders over a period of one semester and arrived at the following conclusions:

1. Although no statistically significant difference was found in the class means on the test for basic mathematical understandings, the data indicated a trend toward the greater achievement by the group using the individually manipulated materials.
2. The use of individually manipulated materials seems to be a somewhat more effective means for building an

understanding than does the teacher demonstration model.

3. A teacher demonstration of instructional materials seemed to promote general mathematical achievement as effectively as does individual manipulation of the materials by the student.

Trueblood (60) compared the technique of student use of materials as opposed to the teacher demonstrating the materials. The experiment was conducted to provide evidence on whether students age 9 to 11 would achieve and retain more by (1) manipulating visual, tactual aids, or (2) observing and telling the teacher how to manipulate such devices. Piaget's stages of intellectual development were used to hypothesize that (1) would be superior to (2).

The pupils taught by observing and telling the teacher how to manipulate the devices scored higher on the post-test than students who manipulated the devices themselves ($p = .10$). There was no significant difference between the two treatments on the retention test.

The role of games in mathematics is discussed in two studies. These games were based on logical skill. Anderson (4) found that a first grade group which used programmed games were superior to a control group on a test which involved problem solving. The experimental group was also superior on retention tests.

Humphrey (37) reported a study which suggested that first graders using active games exhibited greater gains in learning number concepts than students using a workbook to study the same concepts.

Summary

Two points of view seem apparent regarding the issue of learning theories in mathematics.

The Piaget-Bruner point of view places primary emphasis on the process of learning and the importance of discovery and activity experiences in learning. The Gagné-Ausubel point of view places emphasis on the product of learning and the development of a structured sequence of ideas. The Gagné-Ausubel point of view places primary responsibility for instruction with the teacher and the textbook.

The implications for the sequence of curriculum growing from these two positions is quite different. In the Gagné-Ausubel analysis, the highest level of learning is problem solving. Lower levels involve facts, concepts, and principles which must precede the problem solving stage. A learner begins with simple prerequisites and works up, pyramid fashion, to the more complex.

In the Bruner-Piaget analysis, the direction of flow is reversed. Bruner and Piaget have the learner begin with a problem situation. When presented with the problem the learner will move down through the hierarchy and form the needed associations, needed concepts, and finally develop rules for solving the problem.

Dienes (18, p. 47), in discussing the difficulties students have learning mathematics, states,

> The curious fact is that these difficulties have never been systematically or scientifically studied, and consequently the process of learning mathematics is so scanty as hardly to amount to knowledge at all.

More recently, Heimer (30, p. 506) states,

> It seems reasonable to conclude that the extent of substantiative knowledge about construction of efficient instructional sequences in mathematics is at present desperately sparse. Not nearly enough is known about the connection between the logical structure of the knowledge and the psychological processes involved in acquiring the knowledge. Adequate teaching algorithms which specify the steps to be taken in order to construct an instructional sequence in the presence of a given set of educational ends and a given set of circumstances, and with some assurance of efficiency, do not exist.

The theoretical discussion and the research in discovery teaching and learning suffered from a lack of well defined understanding on what is meant by discovery.

Davis (17, p. 59) states,

> There is no agreement on what is meant by either discovery teaching or discovery learning. Nor is there any agreement on what discovery is supposed to accomplish; hence no evidence of its accomplishing or not accomplishing any single objective would change the minds of most who do, or do not believe in it.

The findings of studies which compare expository and discovery methods are ambiguous, and no single study is capable of resolving this pedagogical issue. Only a carefully planned program of research is likely to provide clear answers to the problem of discovery versus expository and guided learning.

In evaluating research in activity learning in mathematics, Kieren (41, p. 516) says,

> Most of the studies were small in scale, and perhaps far too lacking in control and in potential generalizability to be considered good research. Nevertheless, they represent first steps toward answering the complex question of the effect of activity methodologies on the learning of mathematics.

Research results on laboratory learning in mathematics must be incorporated with a theory of mathematics teaching and incorporated into programs of teacher education. Little has been done here.

In light of evidence that elementary school children are in a concrete reasoning stage rather than a formal reasoning stage, most mathematics educators believe that it is desirable to use large amounts of manipulative materials with young children. This same principle would apply to older children who have not yet entered the formal reasoning stage.

Ausubel takes a very clear stand on the role of the laboratory method of instruction. Ausubel (3, p. 338) states,

> The primary responsibility for transmitting the content of a science should be deligated to the teacher and the textbook, whereas primary responsibility for transmitting appreciation of scientific method should be delegated to the laboratory.

Ausubel further states,

> Students waste many valuable hours in the laboratory collecting and manipulating emperical data which, at the very best, helped them rediscover or exemplify

> principles that the instructor could present verbally and demonstrate visually in a matter of minutes. Hence, although laboratory work can be invaluable in giving students some appreciation of the spirit and methods of scientific inquiry, and of promoting problem solving, analytic, and generalizing ability, it is a very time consuming and inefficient practice for routine purposes of teaching subject matter content or illustrating principles when didactic exposition or simple demonstration are perfectly adequate.

In summary then, Ausubel would have teachers divide the labor of instruction. The laboratory would be used to convey the method and spirit of inquiry of the science, whereas the textbook and teacher would assume the burden of transmitting subject matter content.

The major difference between a traditional program and a program built around laboratory situations lies in the role of the child in the learning process.

The former program emphasizes content while the latter emphasizes the experiences of the children in building concepts and strategies.

METHODS AND PROCEDURES

The purpose of this study was to examine the relative effectiveness of a laboratory method of teaching geometry in the sixth grade at Harlan Elementary School, Ames, Iowa. The investigation was in three areas:

1. pupil attitude toward mathematics
2. pupil achievement in geometry
3. pupil achievement on a non-verbal intelligence test

This chapter describes the methods and procedures that were used to gather and analyze the data for this study. This chapter has been divided into five parts: (1) selection of the population for the study, (2) preparation of the materials, (3) class management and experiment execution, (4) testing, and (5) treatment of the data.

Selection of the Population

During the 1969-70 school year, Harlan Elementary School served as a sixth grade center for the Ames Community School District. There were nine sixth grade sections at Harlan. All nine sections were included in the study.

Pupils were assigned to sections by one of two procedures. If the pupil completed fifth grade at Harlan Elementary School he was assigned to section A of the sixth grade. The remaining eight sections of the sixth grade (B, C, D, E, F, G, H, I) were filled with pupils from other elementary centers in the

Ames Community District. Assignments were made so that the eight sections were as heterogeneous as possible.

During summer 1969, the principal of Harlan Elementary School grouped sections B through I. The following guidelines were followed in order to keep these eight sections as heterogeneous as possible.

1. Pupils from other elementary centers were assigned to sections so that all elementary centers were represented in each section.
2. Pupils were assigned to sections so that each section had approximately the same number of males and females.
3. Pupils were assigned to sections on the basis of reading and mathematics achievement.

The intent was to have a range of ability levels present in all sections.

Table 1. Assignment of treatments to sections

	Teacher 1	Teacher 2	Teacher 3
8:45-10:15	B: Control	G: Laboratory-cassette	F: Laboratory
10:15-11:45	-----	H: Control	E: Laboratory-cassette
12:15- 1:45	C: Laboratory	I: Laboratory	-----
1:45- 3:15	A: Laboratory-cassette	-----	D: Control

Three mathematics teachers were assigned to teach the nine sections. Each teacher taught three sections. The assignment of experimental and control sections was made by using a table of random numbers.

Table 1 shows the result of the random assignment of treatments to sections.

Preparation of the Materials

The initial preparation of the laboratory units used in this study began during the fall of 1967. The units were to be flexible enough so that they could be used in upper elementary or junior high school (grades 5-8) mathematics classes.

As was mentioned in Chapter two, the findings of the Madison Project, the writings of Piaget, and the Nuffield Mathematics Project motivated the preparation of laboratory units using an activity approach and an inductive pattern of discovery.

In each laboratory unit the pupil was presented with the opportunity to manipulate physical quantities in order to answer some question about mathematics. The worksheets were designed so that correct responses to questions would lead the pupil to generalize his findings in a rule or formula.

The first twelve laboratory units dealt with geometry and probability. In January, 1968, these units were taught to sixth and eighth grade pupils at Malcom Price Laboratory School, Cedar Falls, Iowa. An informal evaluation of this pilot project was encouraging enough to promote further work on the project.

After the initial try-out at Malcom Price Laboratory School, this writer was invited to present a paper at the Forty-Seventh Annual Meeting of the National Council of Teachers of Mathematics. The title of the paper was "Laboratory Materials and Related Experiences for Grades 5-8." In preparing materials for the presentation a set of laboratory units was developed which focused on geometry.

The final step in the preparation of materials for this study took place after subjects, teachers, length of the study, textbook, and the like were known.

The elementary mathematics series used in the Ames Public Schools during the time of this study was the SRA series (14).

Three units in the textbook dealt exclusively with geometry. The units were:

Unit One - "Rectangles: Area and Perimeter"

Unit Ten - "Measurement of Volume"

Unit Twenty - "Geometry: Circle"

The participating teachers felt that twenty school days would be needed to teach these three units to the control classes and provide time for pre-tests and post-tests.

After the content decision was made and the length of the study determined, twelve additional laboratory units were prepared. Since the treatment groups would not use textbook materials, it was decided that the content presented in all sections should be as alike as possible. The method of presentation is the variable, not the geometric content.

As stated in Chapter two, the Madison Project set down guidelines for the preparation of discovery experiences in mathematics. This researcher used the following guidelines in selecting geometry topics for the laboratory units (16, p. 10).

1. The topic must provide experience with the fundamental concepts and techniques with which the children should become familiar.

2. The topic must provide for active participation by the children.

3. The topic should provide abundant opportunities for the children to make discoveries.

Each laboratory unit was contained in a shoe box. All of the manipulative materials, directions, worksheets, and the like were available to the pupil when he opened the box.

The following eighteen laboratory units were assigned to all students in the experimental classes:

1. Angle Measurement (an exercise using a protractor to measure angles)
2. Square Puzzle (an exercise in arranging seven geometric shapes to form a square)
3. Stellar Polygons (an exercise in geometric constructions using compass and ruler)
4. Curve Stitching (an exercise in geometric construction using ruler, yarn, and needle)

5. Mirror Geometry (an exercise to determine axes of symmetry)
6. Tower of Hanoi (an inductive sequence to establish a formula)
7. Calculation of Pi (π) (an inductive sequence to establish a value for Pi)
8. Volume Relationship (an exercise to establish the volume of a cone and sphere)
9. Area of a Rectangle (an inductive sequence to establish the area of a rectangle)
10. Area of a Right Triangle (an inductive sequence to establish the area of a right triangle)
11. Area of a Parallelogram (an inductive sequence to establish the area of a parallelogram)
12. Area of a Triangle (an inductive sequence to establish the area of a triangle)
13. Area and Perimeter (an exercise to calculate area and perimeter of non-regular geometric shapes)
14. Side-Area Relationships (an inductive sequence to establish a ratio of two measurements)
15. Rectangular Prisms (an inductive sequence to establish the volume of a rectangular prism)
16. Surface Area (an experience in calculating the surface area of a rectangular prism)
17. Construction of Polyhedra (an experience in

constructing models of regular polyhedra)

18. Euler's Formula (an inductive sequence to establish a formula)

The following eight laboratory units were designated as electives:

1. What's My Rule? (a number game which requires inductive thinking)
2. Snowflakes (an exercise in geometric construction using ruler and compass)
3. Polygonal Spirals (an exercise in geometric construction using a ruler)
4. Moebius Strip (an exercise in geometric construction using scissors and moebius strips)
5. How Many Squares? (an exercise in calculating area)
6. Geometric Patterns (an inductive sequence to establish a formula)
7. Roll a Number (an inductive sequence with numbers)
8. Super Detective (an exercise in simple logic)

Appendix A contains pictures of the laboratory units and copies of all the printed material contained in the shoe boxes.

In addition to the basic laboratory unit, the laboratory-cassette treatment group was provided with cassette tapes and cassette players. The cassette tapes were prepared by this researcher. Each cassette tape contained a verbatim reading of the printed materials in the respective laboratory unit.

Pupils in the laboratory-cassette treatment group could read and/or listen to get directions for completing a laboratory unit.

Three complete sets of laboratory units were prepared. One set was placed in each classroom. Three sets of cassette tapes were also prepared. The cassette tapes were stored in a separate shoe box in the classroom. The ten cassette players were moved between rooms as needed.

Approximately one month before the study at Harlan Elementary School, a small pilot study was conducted with a sixth grade class at Gilbert Elementary School, Gilbert, Iowa. The purpose of the pilot was to:

1. Field test the measuring instruments. Readability of items and completion time were examined in light of the pupil's performance in the pilot study.
2. Field test the laboratory units to get information on usability of materials, time needed to complete the laboratory units, and management problems associated with the laboratory units.

Class Management and Experiment Execution

The three control groups were taught using the textbook as the primary source of geometry content. The teachers were encouraged to teach their control classes in a manner typical of their treatment of these same units during the preceding school year. It was agreed that the teachers in control classes would not use any parts of the laboratory units developed by this researcher.

The study lasted twenty school days. The schedule was:

Feb. 24	Day 1	Administer attitude pre-test and the geometry achievement pre-test.
Feb. 25	Day 2	Administer pre-test using Lorge-Thorndike Intelligence Test and complete a laboratory unit.
Feb. 26	Day 3	Administer make-up tests and complete another laboratory unit.
Feb. 27 - March 18	Day 4 - Day 18	Complete one laboratory unit per day. (Optional laboratory units were encouraged after the assigned unit was completed and checked.)
March 19	Day 19	Administer post-test using Lorge-Thorndike Intelligence Test and complete a laboratory unit.
March 20	Day 20	Complete the last laboratory unit and administer the attitude post-test and geometry post-test.

The second day of the study, teachers assigned pupils in the experimental sections to teams. In each section, nine teams of pupils were selected by the teacher. The method used to assign pupils to teams was left to the respective teacher. The only constraint was that the same selection process must be used for both experimental sections. Usually a section was divided into seven or eight teams with three members on a team and one or two teams with four members each. The classes ranged in size from twenty-seven to twenty-nine pupils.

Of the eighteen basic laboratory units, nine were judged to be sequential in nature. For example, the unit on area and perimeter of a rectangle must precede the unit on the area of a right triangle. Because of this constraint, the nine sequential laboratory units were placed in positions ten through eighteen on the assignment list (see Appendix B). The first nine laboratory units did not require a definite order.

After assigning pupils to teams and ordering the laboratory units, a schedule was made. The schedule appears in Appendix B.

The schedule gave pupils their assigned laboratory units from day two through day nineteen.

In addition to the eighteen basic laboratory units, eight optional laboratory units were provided. After pupils completed the experiences in the assigned laboratory unit they were encouraged to complete the optional units. The optional units consisted of mathematical games and constructions. Completion of the optional units could take several hours, hence opportunity was provided for pupils to work on these units over a longer period of time.

The laboratory-cassette teams were assigned to a specific cassette player for the entire study. In this way they became accustomed to the operation of one specific player. Since there were only ten cassette players available, it was necessary to move the cassette players from room to room to accommodate the laboratory-cassette treatment groups.

An answer box was placed in each experimental classroom. This box contained a set of correct answers for all of the worksheets. After all team members had completed the experience in a laboratory unit, one team member would go get the folder containing the answers and the team would evaluate their work. After comparing their worksheet with the one from the answer box, pupils filed their work in individual folders. The teacher could then refer to these folders if they needed information for evaluation.

Once the team had completed its task, it would clean up, replace all parts of the laboratory unit, and then was free to start on an optional laboratory unit.

The school day at Harlan Elementary School was divided into four ninety-minute segments. For the purpose of this study, the ninety-minute segment provided time for study of both mathematics and science. Usually the first forty to fifty minutes of the ninety-minute block were used for the study of mathematics. The remaining time was spent on the study of science.

Basically, the teacher's role in experimental sections was that of a resource person, an advisor, and a source of encouragement. There was no teacher lecture nor use of textbook materials in the experimental sections.

Testing

The testing was in two phases. On day one of the study, the pupils completed the sixty item attitude scale and the twenty-five item geometry achievement test. On day two of the study, the first part of the mathematics period was used to administer the Lorge-Thorndike Intelligence Test.

Make-up tests were scheduled for those pupils who were absent. The make-up tests were completed by the third day of the study. If a pupil did not have all testing completed by the end of the third day he was not included in the study.

The total sixth grade enrollment at Harlan was 251. The pre-test was completed by 235 students.

The post-test was given on day nineteen and day twenty. On day nineteen the Lorge-Thorndike Intelligence Test was administered and on day twenty the attitude scale and geometry achievement test were administered.

Data pertinent to this study was collected by administering six tests to each pupil in the study. Subjects in control and experimental groups were pre-tested and post-tested using:

1. A mathematics attitude test, pre-test and post-test.
2. A geometry achievement test, pre-test and post-test.
3. The Lorge-Thorndike Intelligence Test, pre-test and post-test.

The mathematics attitude test was a compilation of 60 items from the School Mathematics Study Group, National

Longitudinal Study of Mathematical Abilities. This test measured attitudes toward mathematics.

Items for the attitude test were selected from Form 6151 (items 1-14, 36, 37, 40-48) and from Form 6252 (items 1-8, 10, 12-37) of NLSMA Report, No. 1, Part A, X-Population Test Batteries (62).

The geometry achievement test contained twenty-five questions and was composed of items from two sources. The first thirteen items were chosen from the National Longitudinal Study (62). All of the geometry items from the sixth grade level were used in the geometry achievement test. The remaining twelve items were selected from the standardized test series which accompanies the textbook used at Harlan Elementary School. Each item which dealt with geometry was included in the geometry achievement test. This test series is produced by the Greater Cleveland Research Council and is published by Science Research Associates (32).

The first thirteen items for the geometry achievement test were selected from Form 8342 (items 13, 16, 18, 21) and from Form 6262 (items 32-40), Part D of NLSMA Report, No. 1, Part A, X-Population Test Batteries (62).

The last twelve items for the geometry achievement test were selected from the standardized tests which accompany the textbook (32) used at Harlan Elementary School. All the items dealing with geometry were selected to be in the geometry achievement test. The following items were used: Form 6-1B (items 9, 12, 19, 21), Form 6-2B (items 18, 22, 25, 47), Form 6-4B (items 2, 6, 9, 18).

The intelligence pre-test used in this study was the Lorge-Thorndike Intelligence Test, Level 3, Non-Verbal Battery. This test is published in two forms. Form A was used in the pre-test and Form B was used in the post-test (44).

The same form of the attitude test and the geometry test was used in both the pre-test and the post-test.

There were only three pupils in the pre-test who did not complete the post-test battery. This left a total of 232 pupils who were subjects in this study.

All testing was performed in the classrooms with teachers monitoring. Standard IBM type answer sheets were used for all responses. Test scoring and all item analysis was performed by the Testing Service at Iowa State University.

Treatment of Data

The primary goal of this investigation was to evaluate the relative effectiveness of the teacher-textbook method, the laboratory method, and the laboratory-cassette method as measured by post-treatment-tests on attitude toward mathematics, geometry achievement, and non-verbal intelligence. This research also investigated:

1. The effectiveness of the treatments on high, middle, and low I. Q. levels.

2. The effectiveness of the treatments on males versus females.

Table 2 indicates the number of subjects by sex, intelligence group, and treatment group in a three-way classification.

Table 2. Stratification of subjects by treatment, sex, and I. Q. level classification

	Male IQ level			Female IQ level			
Group	Low	Middle	High	Low	Middle	High	Total
Control	9	23	12	8	17	12	81
Lab	13	22	8	7	17	6	73
Lab-cassette	15	19	12	8	12	12	78
Total	37	64	32	23	46	30	232

The statistical model used to analyze the data was analysis of covariance. This technique provided data regarding main effects and interactions. Pupils were statistically equated with respect to the three covariates (pre-test scores on attitude, geometry, and non-verbal intelligence). This analysis assumed that the pupil was the experimental unit.

Since each teacher was assigned a control class, a laboratory class, and a laboratory-cassette class, the teacher effect was not treated as a variable.

The main effects in the analysis of covariance design were: teaching method (treatment), sex of pupils, and I. Q. level of pupils.

The criterion variables were the post-treatment measures of attitude toward mathematics, geometry achievement, and non-verbal intelligence.

The basic model including the effects and sources of variability isolated in the experiment was:

$$Y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\alpha\beta\gamma)_{ijk} + \Delta(X_{ijkl} - \overline{X}_{....}) + e_{ijkl}$$

where

Y_{ijkl} = criterion score on post-test

μ = overall grand mean

α_i = treatment effect
- $i = 1$ for control
- $i = 2$ for laboratory
- $i = 3$ for laboratory-cassette

β_j = effect for sex of pupil
- $j = 1$ for male
- $j = 2$ for female

γ_k = effect for I. Q. level
- $k = 1$ for high I. Q. level
- $k = 2$ for middle I. Q. level
- $k = 3$ for low I. Q. level

$(\alpha\beta)_{ij}$ = 1^{st} order interaction of i^{th} treatment with j^{th} sex of pupils

$(\alpha\gamma)_{ik}$ = 1^{st} order interaction of i^{th} treatment with k^{th} I. Q. level

$(\alpha\beta\gamma)_{ijk}$ = 2^{nd} order interaction of i^{th} treatment and j^{th} sex of pupils and k^{th} I. Q. level

$\Delta(X_{ijkl} - \overline{X}_{....})$ = covariate effect

l = $1,2,\ldots,n_{ijk}$ = individual observation

e_{ijkl} = random error

A second statistical model defined the class as the experimental unit. Class means were used for the nine observations.

The statistical model was:

$$Y_{ij} = \mu + \alpha_i + \beta(\overline{X}_{ij} - \overline{X}_{..}) + e_{ij}$$

where

Y_{ij} = class mean on criterion post-test

μ = overall grand mean

α_i = treatment effect
- $i = 1$ for control
- $i = 2$ for laboratory
- $i = 3$ for laboratory-cassette

$\beta(\overline{X}_{ij} - \overline{X}_{..})$ = covariate effect

j = $1,2,3,\ldots,9$ = observed class mean

e_{ij} = random error

Analysis of the data was completed using standard regression analysis procedures at the Iowa State Computation Center. Calculations were performed on the IBM 360, Model 65 computer.

FINDINGS

Introduction

The findings of this study were based upon the results obtained by testing 232 pupils in nine sixth grade classes at Harlan Elementary School, Ames, Iowa.

To treat the findings in this study, three subdivisions were needed:

1. analysis of the measuring instruments.
2. analysis of covariance on the criterion variables.
3. analysis of significant findings.

The second section above was further subdivided into:

a. analysis of covariance when the experimental unit was the pupil.

b. analysis of covariance when the experimental unit was the class.

Analysis of the Measuring Instruments

Table 3 displays the correlation coefficients of the respective measuring instruments. The correlations of the instruments of a cognitive nature were generally in the .6 to .7 range. The correlation of the attitude instrument with the cognitive type measures was approximately .2.

The correlation of I. Q. test scores with attitude pre-test was .2375, and the correlation of I. Q. test with attitude post-test was only .1993.

Table 3. Correlation of measuring instruments

Test	1	2	3	4	5	6	7
1 I. Q. test	1.0000						
2 Attitude pre-test	.2375	1.0000					
3 Attitude post-test	.1993	.7384	1.0000				
4 Geometry pre-test	.5088	.2200	.1958	1.0000			
5 Geometry post-test	.6150	.2203	.2490	.7269	1.0000		
6 Lorge-Thorndike pre-test[a]	.9440	.2578	.1890	.4612	.5751	1.0000	
7 Lorge-Thorndike post-test[a]	.6158	.2069	.1839	.5019	.6155	.6810	1.0000

[a]Raw score.

The summary in Table 4 indicates that the control sections and the experimental sections were quite similar on I. Q. scores.

The mean I. Q. for the sample was 111.02 with a standard deviation of 10.61. This compares with an expected mean of 100 and expected standard deviation of 15.

The three laboratory sections had a mean I. Q. of 110.41, even though the three sections which made up this treatment had a range of 105.80 to 115.42.

Table 4. Summary of Lorge-Thorndike non-verbal I. Q. pre-test[a] by class section

Classification	Mean	St. deviation
Control sections		
Section 2	111.11	13.64
Section 4	111.22	9.13
Section 8	112.74	9.84
Mean for control	111.69	
Laboratory sections		
Section 3	110.21	8.35
Section 6	105.80	10.01
Section 9	115.42	9.52
Mean for laboratory	110.41	
Laboratory-cassette sections		
Section 1	112.15	10.35
Section 5	110.83	12.76
Section 7	109.67	9.87
Mean for laboratory-cassette	110.88	
Grand mean[b]	111.02	

[a]Seventy-nine items on the test.

[b]Reliability of the test = .85.

The I. Q. score was used to stratify the sample into three sub-samples for the purpose of analyzing interaction effects in the analysis of covariance.

The summary in Table 5 shows attitude gains for all sections except section 7. In this section, the attitude score for the post-test was 1.70 units less than the pre-test score. This loss is not significant, but the lower post-test score indicates that the laboratory-cassette treatment was not

Table 5. Summary of attitude scores for pre-test[a] and post-test

Classification	Mean pre-test	Mean post-test	St. deviation pre-test	St. deviation post-test
Control sections				
Section 2	169.33	172.81	26.26	22.80
Section 4	176.37	176.78	21.31	24.67
Section 8	174.74	179.19	21.62	17.89
Mean for control	173.48	176.26		
Laboratory sections				
Section 3	165.25	170.42	24.96	22.16
Section 6	177.84	184.76	20.92	18.60
Section 9	185.67	187.04	13.90	18.59
Mean for laboratory	176.27	180.79		
Laboratory-cassette sections				
Section 1	179.74	180.96	19.98	21.94
Section 5	166.79	171.00	25.40	27.27
Section 7	177.70	176.00	23.92	27.60
Mean for laboratory-cassette	175.05	176.18		
Grand mean[b]	174.89	177.66		

[a]Sixty items on the test.

[b]Reliability of the test = .94.

effective in creating a more positive attitude toward mathematics in this section.

The reliability for the attitude test was calculated using the Spearman-Brown formula (50, p. 193).

In the summary for the geometry achievement test, Table 6, the average item difficulty was .51. This indicates that approximately 51% of the pupils selected an incorrect response on an average question. The post-test item difficulty was .39 which indicates that approximately 39% of the students selected an incorrect response on an average question. In general, multiple choice items should have a difficulty index in the range .4 to .6.

Table 6. Summary of geometry pre-test[a] and post-test by sections

Classification	Mean pre-test	Mean post-test	St. deviation pre-test	St. deviation post-test
Control sections				
Section 2	11.70	15.33	4.20	4.78
Section 4	12.64	15.85	3.98	4.29
Section 8	12.32	16.22	3.76	3.47
Mean for control	12.25	15.80		
Laboratory sections				
Section 3	12.50	16.00	4.06	4.16
Section 6	11.88	13.40	3.88	5.11
Section 9	12.92	15.83	2.89	3.77
Mean for laboratory	12.47	15.05		
Laboratory-cassette sections				
Section 1	12.85	15.30	2.93	3.93
Section 5	13.00	14.79	4.25	5.30
Section 7	11.15	14.56	4.51	5.21
Mean for laboratory-cassette	12.31	14.88		
Grand mean[b]	12.34	15.26		

[a]Twenty-five items on the test with average item difficulty of: pre-test, .51; post-test, .39.

[b]Reliability (r): pre-test, r = .68; post-test, r = .77.

Table 7. Summary of Lorge-Thorndike pre-test[a] and post-test

Classification	Mean pre-test	Mean post-test	St. deviation pre-test	St. deviation post-test
Control sections				
Section 2	63.81	67.19	9.18	7.91
Section 4	64.81	67.74	5.76	5.95
Section 8	65.52	65.26	7.32	6.40
Mean for control	64.72	66.73		
Laboratory sections				
Section 3	64.17	67.21	5.87	7.14
Section 6	60.36	63.52	9.34	10.04
Section 9	67.63	68.00	5.05	5.12
Mean for laboratory	64.00	66.21		
Laboratory-cassette sections				
Section 1	65.37	69.15	7.40	4.90
Section 5	63.38	65.58	9.26	11.72
Section 7	63.93	67.78	6.68	6.13
Mean for laboratory-cassette	64.26	67.58		
Grand mean[b]	64.34	66.85		

[a]Seventy-nine items on test with average item difficulty of: pre-test, .19; post-test, .16.

[b]Reliability (r): pre-test, r = .74; post-test, r = .70.

Table 7 displays a summary for the Lorge-Thorndike Intelligence Test scores. Raw score, rather than I. Q. score, was used as a covariate because age of student was not treated as a variable in the study.

The item difficulty on the pre-test was .19 and on the post-test was .16. This means that only 19% and 16%

respectively missed an average item on these two tests.

Form A of the Lorge-Thorndike Intelligence Test was used in the pre-test, and Form B was used as a post-test.

Analysis of Covariance

The basic statistical design used in this study treated the pupil as the experimental unit (n = 232). This design enabled the researcher to examine three main effects, first order, and second order interactions.

The main effects were method of class presentation (treatments), sex of pupils, and I. Q. level of pupils.

First order interaction effects between treatments and sex of pupils, treatments and I. Q. level of pupils were studied. The second order interaction between treatments, sex of pupils, and I. Q. level was also studied.

Post-test scores on the attitude toward mathematics test, achievement in geometry test, and non-verbal intelligence test were treated as criterion variables.

The pre-test scores for the criterion variables were used as covariates.

A second set of three tables present the analysis of covariance where the class is treated as the experimental unit (n = 9).

Table 8 displays data on attitude toward mathematics. Null hypotheses 1, 2, 3, and 4 from Chapter one were tested using data from this table. At the .05 level of significance

Table 8. Analysis of covariance: post-test attitude score is the criterion variable (n = 232)[a]

Source of variation	Degrees of freedom	Sum of squares	Mean square	F-value
TREATMENTS	2	125.2658	62.6329	<1
SEX	1	6.5253	6.5253	<1
IQ LEVEL	2	1030.7589	515.3794	2.20
TRTS X SEX	2	343.5088	171.7544	<1
TRTS X IQ	4	1686.1639	421.5410	1.80
TRTS X SEX X IQ	4	2477.5351	619.3838	2.65*
COVARIATE	1	63469.1788	63469.1788	271.17**
ERROR	215	50321.1852	234.0520	

[a]Tabular $F_{4, 200}$ at the .05 level is 2.49. Tabular $F_{2, 200}$ at the .05 level is 3.03.

only null hypothesis 4 was rejected. Null hypothesis 4 states: There will be no significant difference in pupil attitude toward mathematics due to the second order interaction of treatments, sex of pupils, and I. Q. level of pupils, when initial differences between pupils have been adjusted with respect to attitude toward mathematics.

This finding will be discussed in more detail later in this chapter.

The "F" value for the covariate is very large (271.17). Good covariates should account for a significant amount of variation. However, the search for significant "F" values is hampered when such a large percent of the variation is accounted for by the covariate.

Table 9 displays data on attitude when the class was treated as the experimental unit. The treatment effect was not significant at the .05 level (see null hypothesis 13).

Table 9. Analysis of covariance: post-test attitude score is the criterion variable (n = 9)[a]

Source of variation	Degrees of freedom	Sum of squares	Mean square	F-value
TREATMENTS	2	18.1900	9.0950	1.54
ERROR	5	29.6022	5.9204	

[a]Tabular $F_{2, 5}$ at the .05 level is 5.79.

Analysis of covariance for achievement in geometry is presented in Table 10.

The main effect due to I. Q. level of pupils was significant at the .05 level. This finding will be discussed later in this chapter. All of the null hypotheses for the criterion variable of geometry achievement were found to be tenable. As was the case with the attitude covariate, the geometry covariate (F = 154.38) removes a large amount of variation.

Table 10. Analysis of covariance: post-test geometry achievement is the criterion variable (n = 232)[a]

Source of variation	Degrees of freedom	Sum of squares	Mean square	F-value
TREATMENTS	2	2.1027	1.0514	<1
SEX	1	1.2336	1.2336	<1
IQ LEVEL	2	52.5662	26.2831	3.20*
TRTS X SEX	2	19.6998	9.8499	1.20
TRTS X IQ	4	9.5133	2.3783	<1
TRTS X SEX X IQ	4	70.4691	17.6173	2.15
COVARIATE	1	1260.6434	1260.6434	154.38**
ERROR	215	1763.7442	8.2035	

[a]Tabular $F_{4, 200}$ at the .05 level is 2.49. Tabular $F_{2, 200}$ at the .05 level is 3.03.

Achievement in geometry was again treated as the criterion variable in the analysis of covariance in Table 11. In this case, the class was the experimental unit and the

Table 11. Analysis of covariance: post-test geometry achievement is the criterion variable (n = 9)[a]

Source of variation	Degrees of freedom	Sum of squares	Mean square	F-value
TREATMENTS	2	1.577	.789	.826
ERROR	5	4.773	.955	

[a]Tabular $F_{2, 5}$ at the .05 level is 5.79.

treatment effects were not significant at the .05 level.

In the case where non-verbal intelligence test score is considered as the criterion variable, none of the sources of variation were significant at the .05 level. Table 12 displays the covariance analysis. For this summary the experimental unit was the pupil.

Table 12. Analysis of covariance: Lorge-Thorndike post-test score is the criterion variable (n = 232)[a]

Source of variation	Degrees of freedom	Sum of squares	Mean square	F-value
TREATMENTS	2	31.4066	15.7033	<1
SEX	1	47.3962	47.3962	1.62
IQ LEVEL	2	51.9043	25.9526	<1
TRTS X SEX	2	16.6121	8.3060	<1
TRTS X IQ	4	39.2863	9.8216	<1
TRTS X SEX X IQ	4	98.7456	24.6864	<1
COVARIATE	1	1602.9492	1602.9492	54.90**
ERROR	215	6277.7003	29.1986	

[a]Tabular $F_{1, 215}$ at the .05 level is 153.89.

In Table 13, the criterion variable for the analysis was non-verbal intelligence test scores. The treatment effect was not significant at the .05 level. For this summary, the experimental unit was the class.

Table 13. Analysis of covariance: Lorge-Thorndike post-test score is the criterion variable (n = 9)[a]

Source of variation	Degrees of freedom	Sum of squares	Mean square	F-value
TREATMENTS	2	3.949	1.974	.532
ERROR	5	18.562	3.712	

[a]Tabular $F_{2,5}$ at the .05 level is 5.79.

Table 14 displays a summary of "F" values associated with main effects and interaction effects when the experimental unit was the pupil. This is a compilation of data from Table 8, Table 10, and Table 12.

Table 14. Summary of the analysis of covariance "F" values on attitudes, geometry achievement, and Lorge-Thorndike general intelligence for n = 232

Effects	Attitudes	Geometry achievement	Lorge-Thorndike general intelligence
	Classification		
Main effects			
TREATMENTS	.272	.007	.538
SEX	.028	.008	1.620
IQ LEVEL	2.200	3.200*[a]	.889
Interaction effects			
TRTS X SEX	.734	1.200	.285
TRTS X IQ LEVEL	1.800	.154	.336
TRTS X SEX X IQ LEVEL	2.650*[b]	2.150	.846

[a]The 3.200 above compares with a tabular $F_{2, 215}$ at .05 level of 3.04.

[b]The 2.650 above compares with a tabular $F_{4, 215}$ at .05 level of 2.42.

In Table 15, the results from Table 9, Table 11, and Table 13 are summarized. This table displays only the treatment effects where the experimental unit was defined to be the class.

Table 15. Summary of the analysis of covariance "F" values on attitudes, geometry achievement, and Lorge-Thorndike general intelligence for n = 9[a]

Classification	F-value for treatment effect
Attitudes	1.54
Geometry achievement	.83
Lorge-Thorndike general intelligence	.53

[a]Tabular $F_{2, 5}$ at the .05 level is 5.79.

The pre-test means and adjusted post-test means for attitude toward mathematics, geometry achievement, and non-verbal intelligence test scores are presented in Table 16.

The adjusted post-test mean for attitude toward mathematics was highest for the laboratory method. This indicates a tendency for the laboratory method to be more effective.

For geometry achievement the adjusted post-test mean for the control treatment is the highest. This indicates a tendency for the control treatment in this study to be the most effective way to teach geometry.

For the Lorge-Thorndike Non-Verbal intelligence variable, the laboratory-cassette method had the highest post-test adjusted mean. This indicates a tendency for the laboratory-cassette method to be most effective when non-verbal intelligence is measured.

These observations point out trends. No conclusive statements can be made because the findings were not significant at the .05 level.

Table 16. Pre-test and adjusted post-test attitude means, geometry means, and Lorge-Thorndike means

Classification	Pre-test means	Adjusted post-test means
Attitude		
Control	173.48	177.37
Laboratory	176.27	179.71
Laboratory-cassette	175.05	176.05
Geometry		
Control	12.25	15.82
Laboratory	12.47	15.02
Laboratory-cassette	12.31	14.89
Lorge-Thorndike		
Control	64.72	66.72
Laboratory	64.00	66.22
Laboratory-cassette	64.26	67.58

Analysis of Significant Findings

The significant second order interaction in Table 8 requires further analysis.

The multiple "R^2" for the full regression model accounted for 58.55% of the total variance. When the second order interaction was deleted from the regression model, 56.51% of the variation was accounted for. This resulted in a proportion of variance due to the second order interaction of only 2.04%.

Table 17 displays pre-test and post-test means for attitude toward mathematics when the sample was stratified by treatment group and I. Q. level.

Mean differences from pre-test to post-test in the control group was +6.61 for pupils in the low I. Q. level. This compares to +1.73 and 1.53 for the other two treatments.

The laboratory treatment mean differences were +6.96 for the middle I. Q. groups and +6.66 for the low I. Q. groups. The mean difference for the high I. Q. pupils in the

Table 17. Analysis of attitude scores by treatment and I. Q. levels

Classification	Pre-test	Post-test	Difference
Control			
High I. Q.	180.28	182.01	+1.73
Middle I. Q.	172.38	173.92	+1.54
Low I. Q.	165.19	171.80	+6.61
Laboratory			
High I. Q.	183.67	185.25	+1.58
Middle I. Q.	174.97	181.93	+6.96
Low I. Q.	171.15	177.81	+6.66
Laboratory-cassette			
High I. Q.	180.56	179.09	-1.47
Middle I. Q.	173.96	177.20	+3.24
Low I. Q.	169.73	170.09	+0.36

laboratory method was only +1.58.

In the laboratory-cassette treatment the middle I. Q. group means showed a difference of +3.24, and the low I. Q. group means showed a difference of only +0.36. In the high I. Q. level, the group means had a difference of -1.47.

A further breakdown of attitude scores is given in Table 18. This display gives treatment by sex of pupil by I. Q.

Table 18. Analysis of attitude scores by treatments, sex, and I. Q. level

Classification	Pre-test	Post-test	Difference
Control			
High male	179.10	180.17	+1.07
High female	180.54	183.48	+2.94
Middle male	173.80	169.89	-3.91
Middle female	169.10	179.40	+10.30
Low male	169.58	171.92	+2.34
Low female	164.50	167.50	+3.00
Laboratory			
High male	179.92	185.36	+5.44
High female	186.25	182.75	-3.50
Middle male	180.75	181.98	+1.23
Middle female	176.12	182.91	+6.79
Low male	173.01	178.60	+5.59
Low female	169.00	178.00	+9.00
Laboratory-cassette			
High male	171.11	176.78	+5.67
High female	189.00	179.05	-9.95
Middle male	171.55	179.36	+7.81
Middle female	178.33	174.94	-3.39
Low male	172.00	175.14	+3.14
Low female	173.28	169.72	-3.56

level data for pre-test and post-test means. As stated before, the laboratory method of instruction tends to have the most positive effect on pupils' attitudes. Only in the case of the high I. Q. level females was the post-test score less than the pre-test score.

In Table 10, the main effect of I. Q. level for geometry achievement was significant at the .05 level.

The multiple "R^2" for the full regression model dealing with geometry achievement was 62.07%. This means that 62.07% of all the variation was accounted for by the regression equation. When variables for the I. Q. level were deleted from the regression equation, 55.94% of the variation was accounted for. The difference, 6.13%, is a measure of the amount of variation which can be accounted for by the I. Q. level of the pupils.

The results in Table 19 give the pre-test and post-test group means by treatment and I. Q. level of pupils. These findings show the group means for geometry achievement to be directly related to the I. Q. level of the groups. The high I. Q. level groups had highest mean scores. The middle I. Q. level had next highest mean scores, and the low I. Q. group had the lowest mean scores in geometry achievement. This is precisely what would be expected to happen for the I. Q. level main effect.

Table 19. Summary of geometry achievement means for treatments by I. Q. level

	Control method	Laboratory method	Laboratory-cassette method
High I. Q. level			
Pre-test	14.45	15.67	14.35
Post-test	18.82	18.71	17.74
Gain	4.37	3.04	3.21
Middle I. Q. level			
Pre-test	12.45	12.44	13.18
Post-test	15.98	15.00	15.80
Gain	3.53	2.56	2.62
Low I. Q. level			
Pre-test	8.81	10.42	8.94
Post-test	11.63	13.04	10.48
Gain	2.82	2.62	1.54

In summary, only null hypothesis 4 was rejected at the .05 level.

Null Hypothesis 4: There will be no significant difference in pupil attitude toward mathematics due to the second order interaction of treatments, sex of pupils, and I. Q. level of pupils, when initial differences between pupils have been adjusted with respect to attitude toward mathematics.

The relatively small "F" value ($F = 2.65$), the small "R^2" value associated with the second order interaction ($R^2 = 2.04$), and the fact that the pupil was defined as the experimental unit make it imperative to treat this finding as nonconclusive. Plots of the data showed no well defined interaction pattern.

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

Summary

The problem of this investigation was to determine if the use of laboratory materials in teaching geometry to sixth grade pupils at Harlan Elementary School could be established to be significantly more effective than a more traditional method of instruction. Analysis of covariance was used to analyze the data. The criterion variables were post-test scores on tests which measured attitude toward mathematics, geometry achievement, and general non-verbal intelligence. The covariates used were the pre-test scores for the above criterion variables.

Three treatments were randomly assigned to nine sixth grade classes. The treatments were:

1. a laboratory treatment which used laboratory units.
2. a laboratory-cassette treatment which used the laboratory units, cassette players, and cassette tapes.
3. a control treatment which used teacher and textbook.

Fifteen null hypotheses were tested. The first twelve null hypotheses were tested under the assumption that the experimental unit was the pupil. The remaining three null hypotheses were tested under the assumption that the experimental unit was the class.

Treatment, sex of pupil, and I. Q. level of pupil were selected to be main effects in an analysis of covariance design. First order interactions between treatments and sex of pupils, treatments and I. Q. level of pupils were tested. Also, the second order interaction of treatments, sex of pupils, and I. Q. level of pupils was tested.

Null hypotheses were stated for the treatment main effect, first and second order interactions. The four null hypotheses for the criterion variable dealing with attitude toward mathematics were: there will be no significant difference in pupil attitude toward mathematics due to treatments; there will be no significant difference in pupil attitude toward mathematics due to the interaction of treatments and sex of pupils; there will be no significant difference in pupil attitude toward mathematics due to the interaction of treatments and I. Q. level of pupils; there will be no significant difference in pupil attitude toward mathematics due to the interaction of treatments, sex of pupils, and I. Q. level of pupils.

A second set of four null hypotheses was tested using post-test geometry achievement of pupils as the criterion variable, and pre-test geometry achievement as the covariate. A third set of four null hypotheses were tested using non-verbal intelligence post-test scores of pupils as the criterion variable, and pre-test scores as the covariate.

Eleven of the above null hypotheses were found to be tenable. Only in the case of the null hypothesis dealing with attitude toward mathematics due to the second order interaction was the null hypothesis rejected at the .05 level of significance.

The relatively small "F" value (2.65) and small multiple "R^2" (2.04) were interpreted by this researcher as too small to offer any conclusive evidence regarding the null hypothesis.

There was a trend for those students in the middle and low I. Q. levels to have a more positive attitude toward mathematics when the experimental treatments were used.

A second set of null hypotheses was tested with the experimental unit defined to be the class. These hypotheses were: there will be no significant difference in attitude toward mathematics due to treatments; there will be no significant difference in geometry achievement due to treatments; there will be no significant difference in non-verbal intelligence due to treatments. These three hypotheses were found to be tenable.

Implications

The observations contained in this section are the result of discussions with teachers in this study and judgments of the writer. Daily, informal conferences provided an opportunity to compare experiences and discuss the effectiveness of the respective treatments.

Some of these informal observations are supported by data trends, but not at the .05 level of significance.

The following observations seem pertinent:

Using the laboratory method described in this study, the usual procedure was to have pupils perform experiments and collect data. On the basis of these observations, pupils were expected to summarize their findings. This summarization took the form of stating a rule, or predicting what will happen in the next case, or stating some relationship between two different situations.

The above experiences were valuable ones. The pupils in this study were able to complete laboratory units and as a result were able to make judgments regarding geometric concepts. These judgments made the pupil an active participant in the process of discovering relationships which were new to the pupil.

In the control sections, the textbook and teacher provided most of the information, stated the rules, and generally relegated the pupil to the role of a "spectator." The pupil was not involved in processing information, but was held responsible for the facts which resolve from the textbook and from the teacher's presentation.

The accumulation of a set of facts is not categorically bad. Certain basic information is needed in order to be able to solve problems. The issue, however, is that pupils will find it very difficult to solve any new or different problems unless they have the opportunity to learn to process information.

Teachers compared the daily written assignments of control and experimental sections. They could not discern any appreciable differences in quality of the performance between the treatment groups. The geometry achievement in the experimental sections did not differ significantly (.05 level) from the geometry achievement in the control sections.

Teachers did report that most of the pupils in the experimental sections seemed to enjoy the challenge of the discovery lessons.

An additional problem associated with teaching methods which focus on facts and "products" is that educators have no reliable way of knowing which facts and skills will be considered as necessary when the pupil leaves the school setting. In some instances, facts which were dilligently mastered by millions of school children were found to be untrue by the time the children reached adulthood. The laboratory method of instruction used in this study provided pupils with opportunities to process information. Pupils were actively involved in collecting and analyzing data. The teachers were of the opinion that pupils in the laboratory sections exhibited problem solving skills which were not present in the control sections. Pupils were willing to make an "educated guess," they were willing to try a feasible solution and observe the results. This experience is a necessary prerequisite to improving problem solving ability.

The similarity of the laboratory work in mathematics to that of the science laboratory, or to the operation of a business, or to the buying of an automobile should provide for more transfer of learning than the traditional classroom procedure. No attempt was made to measure transfer of learning, but the above is a plausible conjecture.

The observed attitude of both teachers and students toward the experimental classes was quite positive. During the first two days, teachers reported five instances in which students exhibited negative feelings toward the laboratory units. It was of interest to this writer to find that in all of the above cases, the negative feelings came from students who were judged by their teachers to be above average in mathematics ability.

It was not surprising that a few above average pupils resisted a change in teaching methods. Those pupils had experienced success using the teacher-textbook format of instruction. Why should they change?

Usually the more able students were also more verbal. Teaching methods which involved lectures, discussions, and teacher demonstrations appear to be more appropriate for pupils of higher verbal ability.

There was evidence in this study to indicate that pupils in the high I. Q. group responded less favorably to the laboratory method than did the middle and low I. Q. groups (see Table 17). Pupils with high I. Q. can comprehend abstract concepts and manipulate symbols. The need for concrete, manipulative materials is not as great. Requiring these pupils to use laboratory units may retard their learning rate.

By contrast, the child with lower intelligence needs to "see" things happen whenever possible. He cannot comprehend as much from a verbal presentation. Concrete examples are needed, and these examples are most valuable if he is an active participant in generating the data using an activity format. The laboratory method provided a learning experience for pupils who have not mastered the use of abstractions.

Regardless of the verbal ability, those pupils who wait for the teacher to hand out knowledge become dependent learners. The laboratory method enabled pupils to become participants in the learning process. In this way they had a better chance of becoming independent learners.

Typically, the daily, informal evaluation by the teacher reported events in the experimental classes which made it appear that pupils of average and below average ability in mathematics reacted quite positively to the experimental units.

Teachers also reported that boys in the experimental sections seemed more enthusiastic about the laboratory units than were the girls. This observation seems reasonable. Sixth grade boys are usually more receptive to a lesson which requires that they do something of a physical nature. Males tend to perceive themselves as superior to females when mechanical skills are involved. The laboratory units made use of concrete, manipulative materials. Attitude test scores (see Table 18) for males support this observation.

Generally, it was not necessary for the pupil to recall alot of prior learned facts in order to successfully complete a laboratory lesson. The units were designed so that most of the information needed to complete the lesson was developed within the laboratory unit. This made it possible for pupils to achieve success on a particular unit, even though they may not have acquired certain prerequisite learnings which would normally be needed in sixth grade mathematics. For example, pupils with low ability in arithmetic computation made some attractive geometric constructions. One specific example reported by a teacher dealt with a boy who had been having a variety of problems. He was not good in arithmetic, and to compound matters, he was a discipline problem and was held in low esteem by his classmates. However, he was the first member in the class to understand the procedure used to make curve stitchings (see Appendix A, page 106). Most other class members had difficulty with the manipulation of the physical materials in that particular laboratory lesson. This laboratory unit provided an opportunity for him to be the "expert" and give instruction to the other pupils. This single experience had a positive effect on his class attitude as well as improving his relationships with his peers.

Instances such as the above were reported on five occasions. While the number of such reportings was small, experiences which significantly alter one person's behavior may have a positive effect on the entire class. This is especially true if the behavior change occurs in a pupil who has been an acute problem for the teacher.

A majority of the laboratory units used in this study were "open-ended." For example, the unit on stellar polygons outlined the introductory stages in constructing the geometric shape (see Appendix A, page 104). The pupil was provided with an example, but was encouraged to construct shapes of his own. These sixth grade pupils constructed stellar polygons which were original shapes to them and to the writer. As a second example, some of the questions regarding perimeter and area in the construction of snowflakes (see Appendix A, page 153) are difficult for even the most able pupil. In fact, there are questions associated with this unit which may not have solutions.

The first example provided all pupils with a chance to be creative and different, and yet it was not necessary that the pupil be good in mathematics. The second example provided a situation which challenged the most intelligent pupils in the class.

The laboratory method of teaching provided a setting in which pupils were taught individually. All pupils in the experimental sections worked on the laboratory units. Some worked alone, others worked in teams. Pupils working alone completed the assignment at their own pace. The individualization in this case allowed the more able pupil to complete the assigned task in less time.

The laboratory method provided pupils with enrichment experiences. Laboratory units ranged in difficulty from easy to hard. Pupils who completed the easier units in less time were provided with opportunities to explore the more subtle and more difficult ideas associated with the unit. In addition, the optional laboratory units provided enrichment learning.

The laboratory units served as individual learning packages. Typically, low ability pupils are frustrated because just as they first begin to understand a given concept, the class moves to a new topic. This problem was alleviated with the laboratory format. Teachers reported that low ability pupils spent the extra time which was necessary for them to understand the content of the laboratory unit. This was done during study time, before and after school.

The laboratory lessons, like any other lessons, required careful planning to be successful. However, laboratory lessons were more difficult to prepare than textbook lessons.

There were several reasons for this. First, the lessons required the use of manipulative materials which were not present in the classroom. Advanced planning was needed in order that the writer and the teacher had time to collect, or have pupils collect, the needed materials. Second, the lessons were developed independent of the textbook. For this study there was very little available from commercial sources and the mathematics classrooms used in the study had little, if any, laboratory materials. The result, then, is that the teachers and the writer prepared all of the laboratory units. Journals and textbooks provided guidance and suggestions, but no more. Third, the classroom in the study had a more permissive atmosphere. As the pupils first opened the laboratory unit, there was time for "free play" and nonproductive exploration. In addition, some of the laboratory units were game type experiences and resulted in more noise and exuberance on the part of some students. This meant that the teachers expected more activity, noise, and movement on the part of the pupils.

Whenever a pupil finished the assigned laboratory unit in an experimental section he would go pick out an optional laboratory unit, or play a mathematical type game, or wonder about the room and look at what other pupils were doing. In those classes which used cassette tapes and players, the students who had finished their assigned unit could not only see what other students were doing, but they could listen to the cassette player and hear directions and questions associated with the unit.

The teachers felt that those pupils who choose to wonder about the classroom and observe other pupils at work were putting their time to profitable use. Pupils knew which laboratory units were assigned for the next two or three days, and it was possible for them to locate those laboratory units and observe how other pupils were working on the unit.

The situation described above could help some students by the fact that they get an introduction, prior to the time that they are assigned the unit.

There could also be detrimental side effects. If pupils watch other pupils work with inductive sequences, some of the benefits of a laboratory experience may be removed. One intent of a laboratory type experience is to make the pupil a "participant" rather than a "spectator." An important contribution of an inductive sequence is to make it possible for pupils to generalize rules from personal observations. If pupils observe other students discover rules prior to their exposure to the same laboratory unit, some important benefits may be lost.

The use of cassette tapes and players had very brief, early, motivational effects. The results of the study tend to show that the laboratory-cassette groups generally had less favorable adjusted post-test means in attitude toward mathematics and geometry achievement. It may be more efficient to provide cassette tapes and players on an optional basis rather than making the use of such media part of each laboratory experience.

The laboratory method used in this study provided little opportunity for pupils to do drill and practice. The laboratory units were developmental in nature. The intent was to provide the pupil with activity experiences which would enable him to generalize rules and formulas. No provision was made for drill to reinforce the generalization. Normally, teachers would provide review experiences and drill routines after the development of an idea. Because teacher guidance and lecture was minimized in the case of the experimental classes, the reinforcement of initial learning seemed to be lacking.

In the judgment of this researcher, too much time was spent gathering data. The total pre-test required that the pupils work on five separate answer sheets. The tests took approximately two hours to administer. This was done over a two day period. Pupils seemed to resent the number of tests they were asked to take. Shorter tests, with more difficult items might be a better format for elementary pupils.

In conclusion, even though the findings were not significant at the .05 level, the teachers and the writer share the opinion that the experimental sections displayed a more positive attitude toward mathematics than those pupils in the control sections. In addition, the laboratory method was judged: to provide more opportunities for individualizing instruction; to provide more opportunities for enrichment; to provide more opportunities for the pupil to discover relationships and process information; and to be more effective with pupils in the middle and low I. Q. levels. If the above judgments are valid, it must be concluded that the measuring instruments used in the study were not sensitive enough or not appropriate to measure the observations discussed in this section.

Limitations

This study was limited to 232 sixth grade students at Harlan Elementary School in Ames, Iowa.

The experimental methods were limited to mathematics. It would not be proper to apply these findings to any other content field, nor would it be proper to generalize these findings to other grade levels.

The classrooms used in the study were not designed for a laboratory approach to teaching. The rooms were rather crowded and table space was at a minimum. In one classroom there were no tables available for student use. The laboratory units had to be set upon a cluster of three or four pupil desks which were pushed together to form a larger surface.

The post-tests were given on days when the local high school team was playing in the state basketball tournament. Several of the pupils in the study seemed rather upset because they were not excused from school to attend the game.

The geometry achievement test contained twelve items from a test battery produced by the publishers of the textbook used in the control class. There were no items on the test which were written by this researcher. The geometry achievement test contained no items which were designed to measure any content unique to the laboratory units. If the geometry achievement test was biased, it seems reasonable to conclude that it was biased in favor of the control groups.

Standard I.B.M. type answer sheets were used to collect all data. The answer sheet used for the attitude scale had a bigger space between responses 4 and 5 than between any other two successive responses. This spacing may have caused pupils to avoid response 5 because it was not in the same pattern as responses 0 through 4.

A second problem with answer sheets occurred because students had to fill out three answer sheets, one for each part, in taking the Lorge-Thorndike Intelligence Test. Each part of the test took nine minutes and it took that long to collect answer sheets, distribute another set, and complete all the information prior to beginning another part of the test. It would have been better to purchase the answer sheets prepared by the test publishers.

The teacher variable was not present in the statistical analysis. Since each teacher taught one class using each treatment, the teacher variable was not analyzed. It was assumed to be insignificant.

This researcher attempted to minimize the "Hawthorne effect" by having teachers inform both control and experimental classes that they were a part of the study.

Conclusions

As stated previously, the problem of this study was to answer five questions and test fifteen null hypotheses. The first question was: Can laboratory units be developed which teach the basic geometry content in a sixth grade mathematics program? The textbook used in the control classes focused primarily on metric geometry. The laboratory format can be used to teach concepts dealing with measure of length, area, and volume. Students taught by the laboratory method did as well on the geometry achievement post-test as those students who had the teacher-textbook method in the control classes.

The second question was: Can a laboratory method work effectively and efficiently in a sixth grade classroom? Teachers reported that students in the experimental classes adjusted quite well to the new method. By the second or third day students were able to get the assigned shoe box, locate a cassette player and the appropriate tape, and begin their investigations. There were no instances of materials being lost or destroyed. One of the cassette players malfunctioned and was replaced.

The third question was: Can teachers effectively use a laboratory method without the benefit of specific, formal preservice or inservice training? The teachers adjusted very quickly to the laboratory method. The teachers suggested improvements in the laboratory units, and all three teachers plan to use some of the units in subsequent years. After the termination of the study, the teachers used the laboratory units in the control classes. The teachers in this study had no difficulty adjusting to a laboratory method of teaching.

The fourth question was: Can pupils effectively make the transition to an activity type mathematics curriculum using laboratory units? The pupils adjusted quite well to the new format. Some early complaints came from a few of the more able students. They could not see why they had to "do it this way." These complaints were not heard after the third or fourth day. In general the students adjusted quite well to the experimental method.

The fifth question was: Can pupils effectively use cassette tapes and cassette players to obtain directions and information to complete laboratory units? The pupils in the laboratory-cassette groups had little or no difficulty in operating the cassette players. There were no serious problems with malfunction, and no materials were lost or damaged.

The fifteen null hypotheses were related to three criterion variables (attitude toward mathematics, geometry achievement, and non-verbal intelligence), and the effects of teaching method. In addition, the interactions of teaching method, sex of pupil, and I. Q. level of pupil were of interest.

On the basis of the findings in this investigation, the following conclusions seem reasonable:

1. Pupils in the experimental classes did as well on the geometry achievement tests as pupils in classes which used the teacher-textbook method.
2. The laboratory methods used in this study did not significantly affect pupils' attitudes toward mathematics. However, there appears to be a trend for the laboratory method to be more effective with pupils in the middle and low I. Q. levels.
3. Laboratory methods of teaching sixth grade mathematics can be used by teachers without prior in-service or preservice training.
4. Classrooms can be modified to accommodate a laboratory method of teaching elementary mathematics.

Recommendations for Further Research

This experiment could be replicated to validate the findings. However, the writer feels that the following changes in the experiment would make the findings more significant and of greater research value.

1. Use a laboratory method to teach a full year of mathematics in the sixth grade.
2. Use a test of critical thinking and a test of scientific method as criterion variables. In this way a researcher could evaluate the effectiveness of the laboratory method as it relates to a pupil's ability to think critically and process information.
3. Vary the selection of treatments. This could be done in a variety of ways. Three examples are:
   a. Use the laboratory method as one treatment and as a second treatment have the teacher demonstrate the laboratory units.
   b. Use the laboratory method as one treatment and as a second treatment use film loops, pictures, closed circuit television, and other media to demonstrate the activity in each laboratory unit.
   c. Use the laboratory method as one treatment, and as a second treatment combine laboratory units with a teacher-lecture method.
4. Design a longitudinal study of a two or three year duration.
5. Include a retention test to get additional data on the criterion variables.
6. Design laboratory studies at different grade levels. Is there an optimal age to introduce laboratory work?

7. Provide laboratory experiences in more than one content field. Team teach science and mathematics using a laboratory method. The impact of a laboratory method in only one subject may be lost because all other subjects use the teacher-textbook method.
8. Design a study in which the use of laboratory units is optional with the pupils or at the suggestion of the teacher. In this way, the activity experiences with physical materials could be provided whenever the pupil or the teacher felt that it could be beneficial.

Above all, ongoing, cooperative efforts of many researchers are needed to seek clear answers to the many questions in mathematics education. The "one-shot" studies of the doctoral student cannot provide the long-term, indepth studies of learning and teaching which are sorely needed.

BIBLIOGRAPHY

1. Ausubel, D. P. Facilitating meaningful verbal reception in the classroom. The Arithmetic Teacher 15:126-132. 1968.

2. Ausubel, D. P. The psychology of meaningful verbal learning. New York, N.Y., Grune and Stratton. 1963.

3. Ausubel, D. P. and F. G. Robinson. School learning. New York, N.Y., Holt Rinehart and Winston, Inc. 1969.

4. Anderson, R. C. Can first graders learn an advanced problem skill? Journal of Educational Psychology 56:283-94. 1965.

5. Beckland, L. A. Independent study: an investigation of the effectiveness of independent study of novel mathematical materials in the elementary school. Unpublished Ph.D. thesis. Minneapolis, Minn., Library, Univ. of Minnesota. 1968.

6. Begle, E. G. and H. G. Richey. Mathematics education. The Sixty-ninth Yearbook of the National Society for the Study of Education, Part 1. Chicago, Ill., University of Chicago Press. 1970.

7. Biggs, E. E. and J. R. MacLean. Freedom to learn. Reading, Mass., Addison-Wesley Ltd. 1969.

8. Bruner, J. S. Some elements of discovery. In Schulman, L. and E. Keislan, eds. Learning by discovery: a critical appraisal. Pp. 101-113. Chicago, Ill., Rand McNally. 1966.

9. Bruner, J. S. Toward a theory of instruction. Cambridge, Mass., Belknap Press. 1966.

10. Clarkson, D. M. Mathematical activity. The Arithmetic Teacher 15:493-97. 1968.

11. Copeland, R. W. How children learn mathematics: teaching implications of Piaget's research. New York, N.Y., The Macmillan Co. 1970.

12. Craig, R. C. Directed versus independent study of established relations. Journal of Educational Psychology 47:223-234. 1956.

13. Cronbach, L. J. Learning by discovery: a critical appraisal. Chicago, Ill., Rand McNally and Co. 1966.

14. Cunningham, G. S. and R. P. Fisher. Greater Cleveland mathematics program. Grade 6. Chicago, Ill., Science Research Associates, Inc. 1968.

15. Davidson, P. J. and A. W. Fair. A mathematics laboratory--from dream to reality. The Arithmetic Teacher 17:105-111. 1970.

16. Davis, R. B. Discovery in mathematics. Menlo Park, Calif., Addison-Wesley Publishing Co. 1964.

17. Davis, R. B. Discovery in the teaching of mathematics. In Schulman, L. and E. Keislar, eds. Learning by discovery: a critical appraisal. Pp. 118-119. Chicago, Ill., Rand McNally. 1966.

18. Dienes, F. P. An experimental study of mathematics learning. London, England, Hutchinson and Co., Ltd. 1963.

19. Dienes, F. P. Some basic processes involved in mathematics learning. In Scandura, J., ed. Research in mathematics education. Pp. 21-34. Washington, D.C., National Council of Teachers of Mathematics. 1967.

20. Dienes, F. P. and E. W. Golding. Modern mathematics for young children. New York, N.Y., Herder and Herder. 1966.

21. Fisher, J. R. An investigation of three approaches to the teaching of mathematics in the elementary school. Unpublished Ph.D. thesis. Pittsburgh, Pa., Library, Univ. of Pittsburgh. 1968.

22. Fitzgerald, W. M. Self-selected mathematics learning activities. U.S. Office of Education Cooperative Research Project No. 2047. 1964.

23. Fitzgerald, W. M. A mathematics laboratory for prospective elementary school teachers. The Arithmetic Teacher 15:547-551. 1968.

24. Gagné, R. M. Learning hierarchies. Educational Psychologist 6, No. 1: 3-6. 1968.

25. Gagné, R. M. Studies in personnel and industrial psychology. Homewood, Ill., The Dorsey Press. 1967

26. Gibb, E. G. Through the years: individualizing instruction in mathematics. The Arithmetic Teacher 17:396-401. 1970.

27. Glennon, V. J. and L. G. Callahan. Elementary school mathematics: a guide to current research. Third edition. Washington, D.C., Association for Supervision and Curriculum Development, NEA. 1968.

28. Great Britain Central Advisory Council for Education. Children and their primary schools. London, England, Her Majesty's Stationary Office. 1967.

29. Guthrie, J. T. Expository instruction vs. the discovery method. Journal of Educational Psychology 58:45-49. 1967.

30. Heimer, R. T. Conditions of learning in mathematics: sequence theory development. Review of Educational Research 39:506. October, 1969.

31. Henderson, K. B. Research on teaching secondary school mathematics. Handbook of Research on Teaching. Chicago, Ill., Rand McNally. 1963.

32. Ho, Wai-Ching and Sarita Schrock. Greater Cleveland mathematics tests. Chicago, Ill., Science Research Associates, Inc. 1969.

33. Holt, J. How children fail. New York, N.Y., Dell Publishing Co., Inc. 1964.

34. Hooten, J. R. Proceedings of national conference on needed research in mathematics education. Journal of Research and Development in Education. Vol. 1, No. 1. Athens, Georgia, Univ. of Georgia. 1967.

35. Howes, V. M. Individualizing instruction in science and mathematics. New York, N.Y., The Macmillan Co. 1970.

36. Hudgins, B. D. The effects of initial group experience upon subsequent individual ability to solve arithmetic problems. Dissertation Abstracts 22:2851-52. 1958.

37. Humphrey, J. H. An exploratory study of active games in learning of number concepts by first grade boys and girls. Perceptual and Motor Skills 23:341-342. 1965.

38. Kersh, B. Y. The adequacy of "meaning" as an explanation for the superiority of learning by independent discovery. Journal of Educational Psychology 49:282-292. 1958.

39. Kersh, B. Y. Directed discovery vs. programmed instruction: a test of a theoretical position involving educational technology. In Della-Piana, G. M. Sequence characteristics of text materials and transfer of learning, part 1. Pp. 2.23-2.24. Salt Lake, Utah, University of Utah Bureau of Educational Research. 1965.

40. Kersh, B. Y. The motivated effect of learning by directed discovery. Journal of Educational Psychology 53:65-71. 1963.

41. Kieren, T. E. Activity learning. Review of Educational Research 39:516. October, 1969.

42. Kittell, J. E. An experimental study on the effect of external direction during learning on transfer and retention of principles. Journal of Educational Psychology 48:391-405. 1957.

43. Leeb-Lundberg, K. Kindergarten mathematics laboratory: nineteenth-century fashion. The Arithmetic Teacher 17: 372-386. 1970.

44. Lorge, I. and R. L. Thorndike. The Lorge-Thorndike intelligence test, level 3, form a, nonverbal battery. Boston, Mass., Houghton Mifflin Co. 1954.

45. Matthews, G. The Nuffield mathematics teaching project. The Arithmetic Teacher 15:101-102. 1968.

46. May, L. J. Learning laboratories in elementary schools in Winnetka. The Arithmetic Teacher 15:501-506. 1968.

47. Moody, W. B. An investigation of the relationship between fifth-grade student and teacher performance on selected tasks involving nonmetric geometry. Unpublished Ph.D. thesis. College Park, Maryland, Library, Univ. of Maryland. 1968.

48. Morrison, H. C. American schools: a critical study of our school system. Chicago, Ill., The Univ. of Chicago Press. 1943.

49. The Nuffield Foundation. Nuffield Mathematics Project. I do, and I understand. New York, N.Y., John Wiley and Sons, Inc. 1967.

50. Nunnally, J. C. Psychometric theory. New York, N.Y., McGraw-Hill Book Co., Inc. 1967.

51. Phillips, J. L. The origins of intellect: Piaget's theory. San Francisco, Calif., W. H. Freeman and Co. 1969.

52. Piaget, J. The origins of intelligence in children. New York, N.Y., International Universities Press. 1952.

53. Price, J. Discovery: its effect on critical thinking and achievement in mathematics. The Mathematics Teacher 60:874-876. 1967.

54. Scandura, J. M. An analysis of exposition and discovery methods of problem solving instruction. The Journal of Experimental Education 33:149-159. 1964.

55. Schulman, L. S. and E. R. Keislar. Learning by discovery: a critical appraisal. Chicago, Ill., Rand McNally. 1966.

56. Scott, L. Trends in elementary school mathematics. Chicago, Ill., Rand McNally Co. 1966.

57. Snyder, H. D. A comparative study of two self-selection-pacing approaches to individualizing instruction in junior high mathematics. Dissertation Abstracts 28:159A. 1967.

58. Stacey, C. L. The law of effect in retained situations with meaningful material. Univ. of Minnesota Studies in Education, No. 2:74-103. Minneapolis, Minn., The University of Minnesota Press. 1949.

59. Toneym, J. S. The effectiveness of individual manipulation of instructional materials as compared to a teacher demonstration in developing understanding in mathematics. Dissertation Abstracts 29:1831A. 1968.

60. Trueblood, C. R. A comparison of two techniques for using visual-tactual devices to teach exponents and non decimal bases in elementary school mathematics. Dissertation Abstracts 29:190A. 1968.

61. Wills, H. Transfer of problem solving ability gained through learning by discovery. Unpublished Ph.D. thesis. Urbana, Ill., Library, Univ. of Illinois. 1967.

62. Wilson, J. W., L. S. Cahen, and E. G. Beagle. NLSMA reports. No. 1, part a, x-population test batteries. Stanford, Calif., Leland Stanford Junior University. 1968.

63. Wingo, M. Theoretical basis for activities programs. In Herrick, V. E. and R. W. Tyler. Toward improved curriculum theory. Supplementary Education Monograph No. 71. Chicago, Ill., University of Chicago Press. 1950.

64. Worthen, B. R. A study of discovery and expository presentation: implications for teaching. Journal of Teacher Education 19:223-242. Summer, 1968.

APPENDIX A

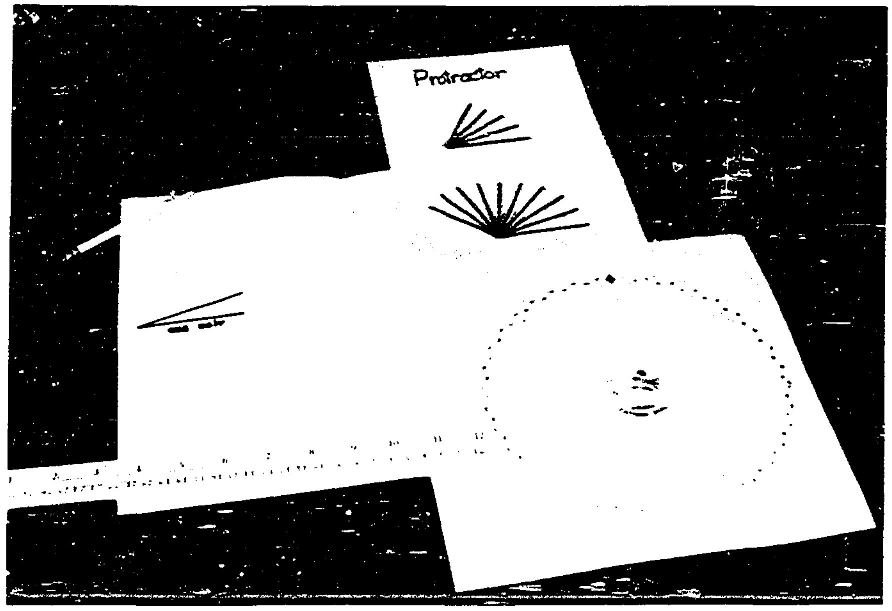


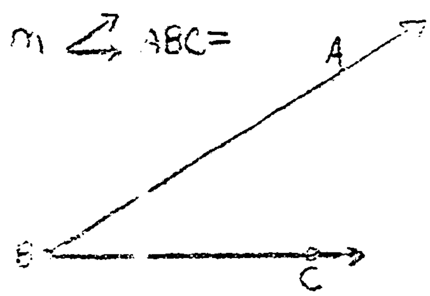
Figure 1. Angle measurement

Pupils will assign numbers to angles using the protractors provided. The basic unit is given. The pupils must locate an origin, then move the unit to measure the interior of the angle.

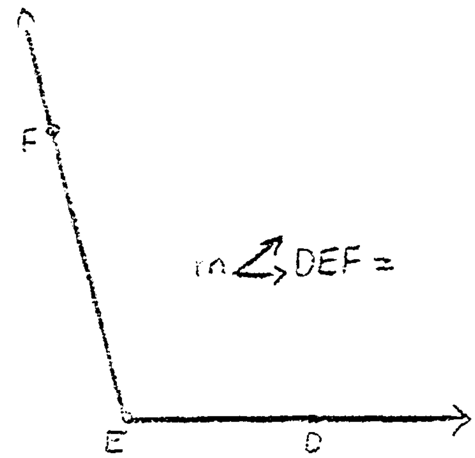
MEASURING ANGLES

In the large envelope are protractors which you will use to assign numbers to angles. The number represents the measure of the angle. Measure the following angles.

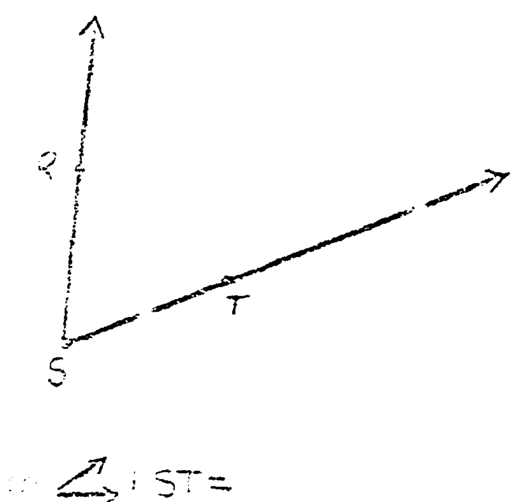
a)



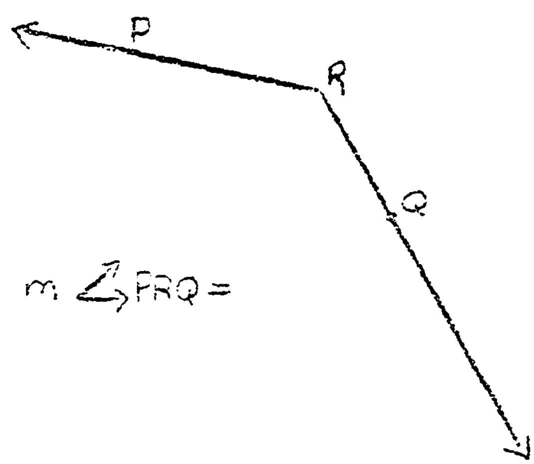
b)



c)



d)



Use the protractors to draw angles whose measure is:

a) m ∠ ABC = 1 unit

b) m ∠ DBF = 3 units

c) m ∠ GIB = 8 units

d) m ∠ XYZ = 6 units

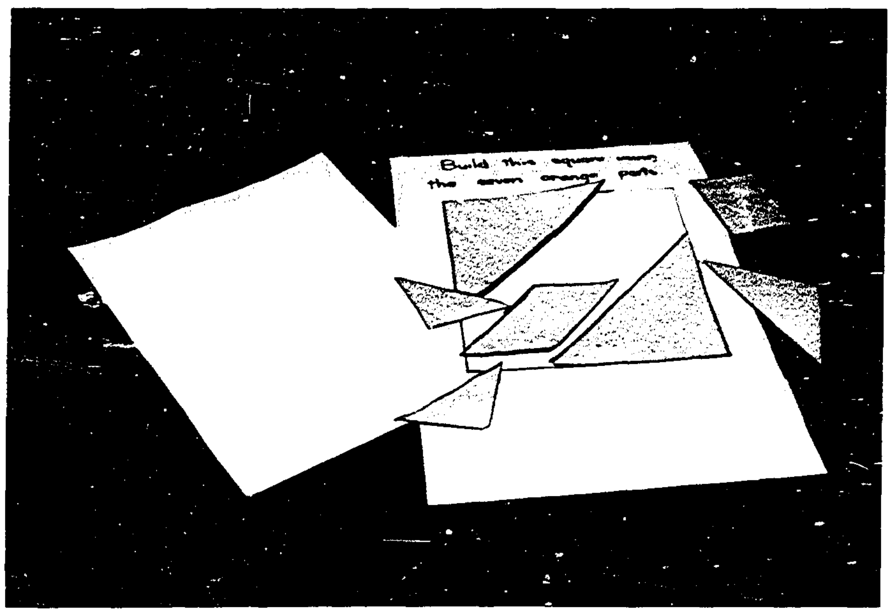


Figure 2. Square puzzle

This unit requires that the pupils manipulate geometric figures. The square puzzle should be an enjoyable experience for pupils. If they become too frustrated, give them some help.

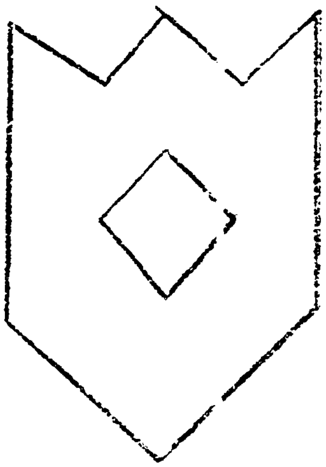
SQUARE PUZZLE

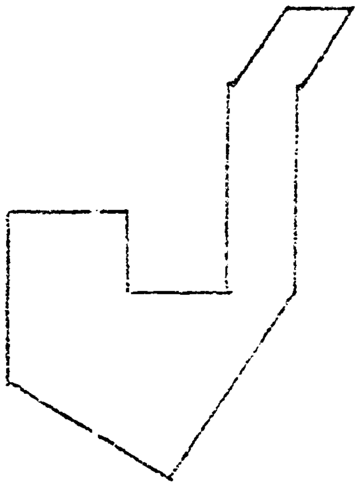
103

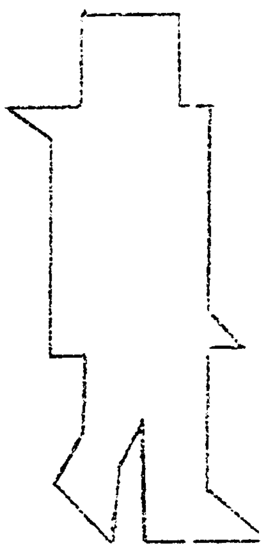
The object of this shoe box is to see if you can fit all 7 pieces of this puzzle into the square.

All pieces must be used and they should fit exactly. The dark line around each peice of the puzzle is the "up side."

After you solve this square puzzle, you might enjoy trying to make some of the following puzzle shapes. These can be made using the same 7 parts that were used to solve the square puzzle.







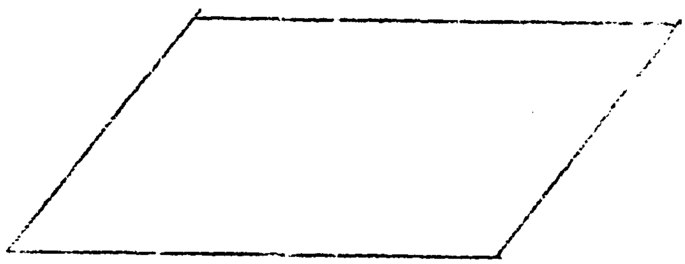
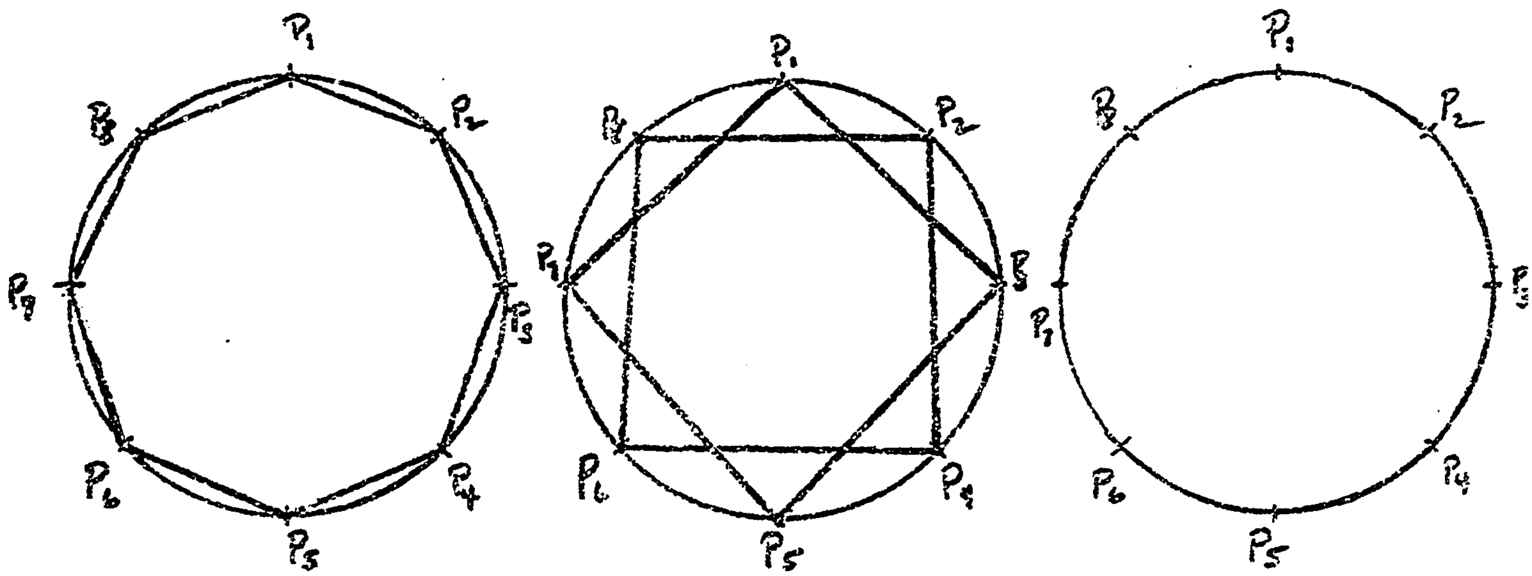


Figure 3. Stellar polygons

The basic objective is to demonstrate to the pupil that neatness and precision are important. If the pupil is neat and uses good quality drawing instruments, his drawing will look nice. Do not let the $\overline{P_1P_2}U$ $\overline{P_2P_3}U...$ notation confuse the students. The intent of this unit is to construct stellar polygons, and the notation is of much less importance at this time. Start the pupils with the octagon.

CONSTRUCTION OF STELLAR POLYGONS



Two stellar polygons have already been completed. In the first one we have joined the successive points, P_1 to P_2 to P_3 etc. This is more formally stated as $\overline{P_1P_2} \cup \overline{P_2P_3} \cup \overline{P_3P_4} \cup \overline{P_4P_5} \cup \overline{P_5P_6} \cup \overline{P_6P_7} \cup \overline{P_7P_8} \cup \overline{P_8P_1}$.

In the second figure we have $\overline{P_1P_3} \cup \overline{P_3P_5} \cup \overline{P_5P_7} \cup \overline{P_7P_1}$. Now that we are back to P_1 with points left, we continue with $\overline{P_2P_4} \cup \overline{P_4P_6} \cup \overline{P_6P_8} \cup \overline{P_8P_2}$. This completes the second stellar polygon.

In the third figure, continue to omit points. In the first case we have joined successive points. In case 2 we omitted one point. Continue to case 3 and omit 2 points. For example, draw $\overline{P_1P_4} \cup \overline{P_4P_7} \cup \ldots.$

In case 4 omit 3 points.

In case 5 omit 4 points.

You now have 5 stellar polygons using a circle divided into 8 congruent parts. You may wish to construct a circle with 16 arcs. See how many different stellar polygons you can make. If you wish you can inscribe a second stellar polygon inside the original one. Color these if you wish. It is also simple to construct a circle with 6, 12, and 24 congruent areas.

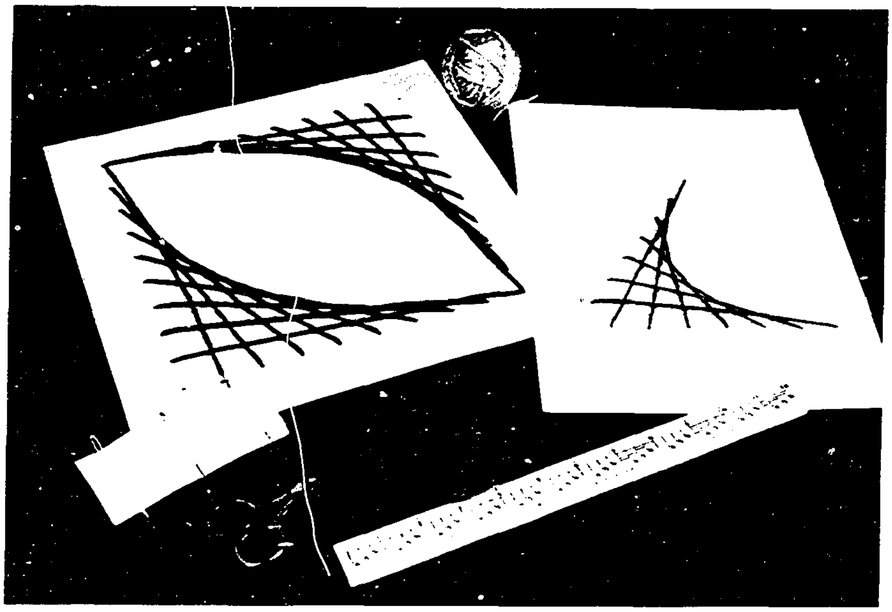
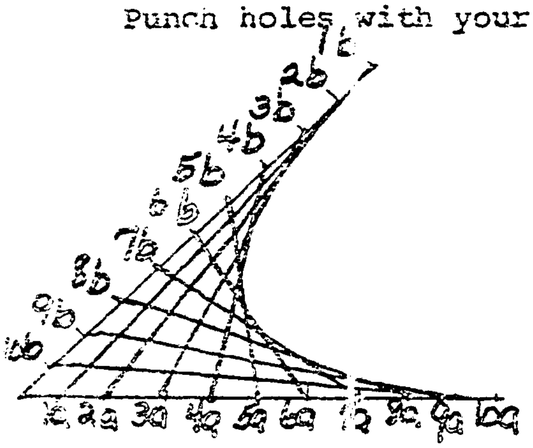


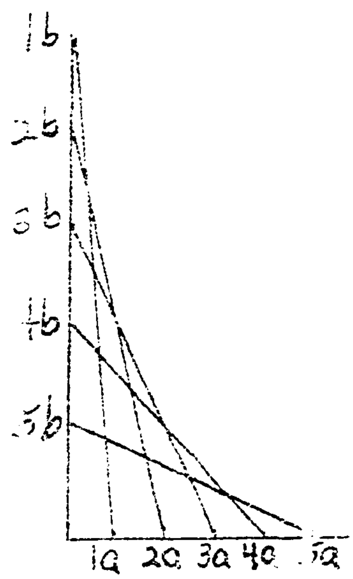
Figure 4. Curve stitching

A measurement activity. Students should be encouraged to be precise, and above all show some imagination in making new designs. Pupils should make three or four designs.

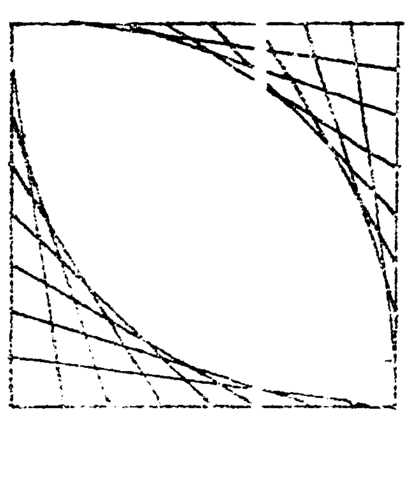
You will need a large needle, some thin colored wool or string, and some pieces of cardboard. Draw two lines which intersect to form an angle as shown below. Mark off each line in ½ inch units. Punch holes with your needle at these marks. Number the marks as shown. With your needle and wool, come up through 1a, go down through 1b. Go up through 2b and down through 2a, up 3a, down 3b, etc. The long stitches will be on the front of the card and the short stitches on the back.



Try this again with the right angle. On one of the lines, mark off ½ inch units and on the other use ¼ inch units. Then number and stitch as before.



Start with the square and make this pattern.



Now make some curve stitching of your own.

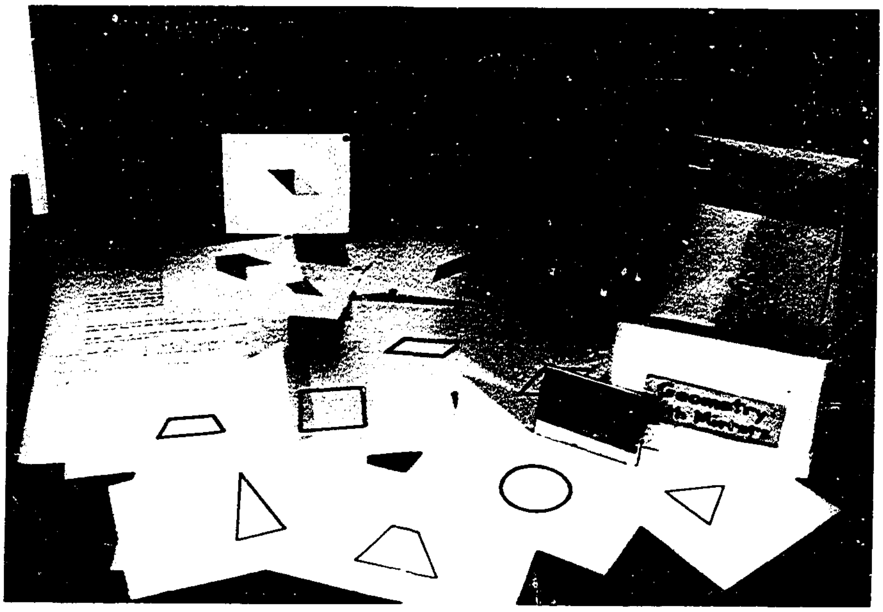


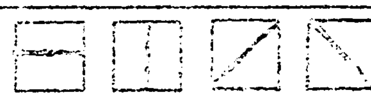
Figure 5. Mirror geometry

Pupils will sketch axes of symmetry for the geometric shapes in Set II. Set I is to acquaint the pupils with positioning the mirror and cards.

Open the envelope containing Set 1. Take a glass mirror and a set of cards. Look at the card with the green dot. Using the mirror and the card with the <u>green dot</u>, try to make a shape which is exactly like the shape on one of the cards with a red dot. Some are easy; some are not. Set the mirror on its edge on the card with the green dot.

After you have practiced with Set 1, you will replace these cards (be careful to get them back into the original groups).

Open Set 2. With Set 2, I want you to record all the ways you get the same shape by using the mirror. I found 4 ways to get a square. The green line represents the mirror.

Geometric Figure	Mirror Positions
SQUARE	
RECTANGLE	
CIRCLE	
PARALLELOGRAM	
RHOMBUS	
ISOSCELES TRIANGLE	
EQUILATERAL TRIANGLE	
ISOSCELES TRAPEZOID	
TRAPEZOID	
REGULAR HEXAGON	

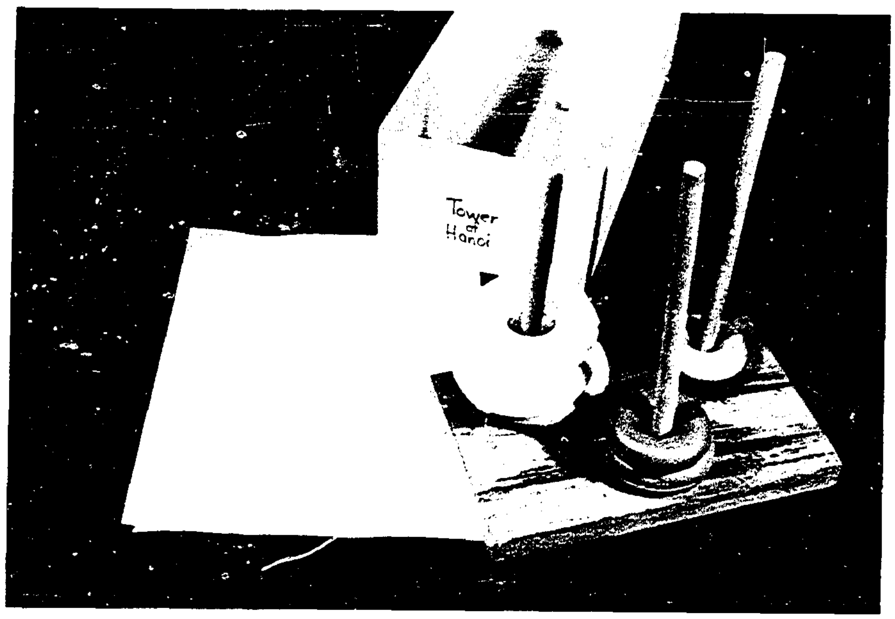


Figure 6. Tower of Hanoi

This unit requires that the pupil generalize an inductive sequence to state a rule. Allow some time for free play before counting the number of moves.

First make the set up look like Figure 1.

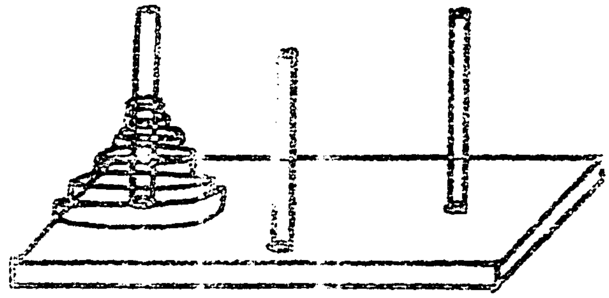


Figure 1.

There are 2 problems in this shoe box.

PROBLEM 1: Move the rings from one peg to another. There are two rules to follow:

a) Move only one ring at a time.

b) You cannot place a larger ring on a smaller one.

Now see if you can get all the rings moved from one peg to either one of the other two.

PROBLEM 2: What is the fewest number of moves needed to move a given number of rings?

To answer this problem start with 1 ring. It takes only 1 move (see Figure 2).

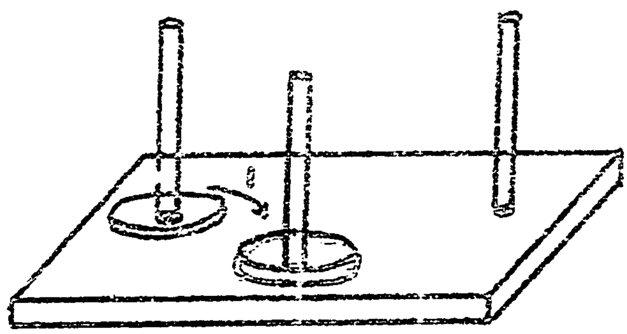


Figure 2.

If n = 1, then M = 1.

n = number of rings

M = number of moves

Now try two rings. The minimum number of moves (M) is 3. Look at Figure 3 and try it yourself.

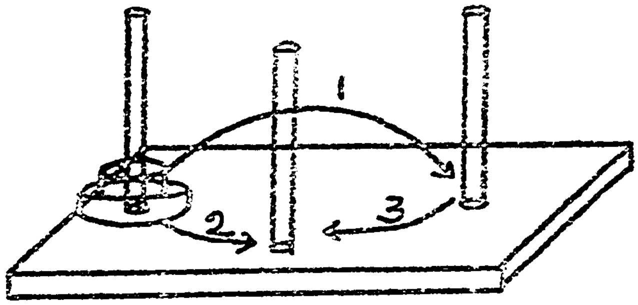


Figure 3.

If n = 2, then M = 3.

Now complete the table. First do 3 rings, then 4, 5, 6.

n	M
1	1
2	3
3	
4	
5	
6	
7	
10	

Look for a pattern. How many moves for 7 rings? for 10 rings? Write a formula for finding M (the minimum number of moves) given n rings.

Pattern: ______________________________

____

____

____

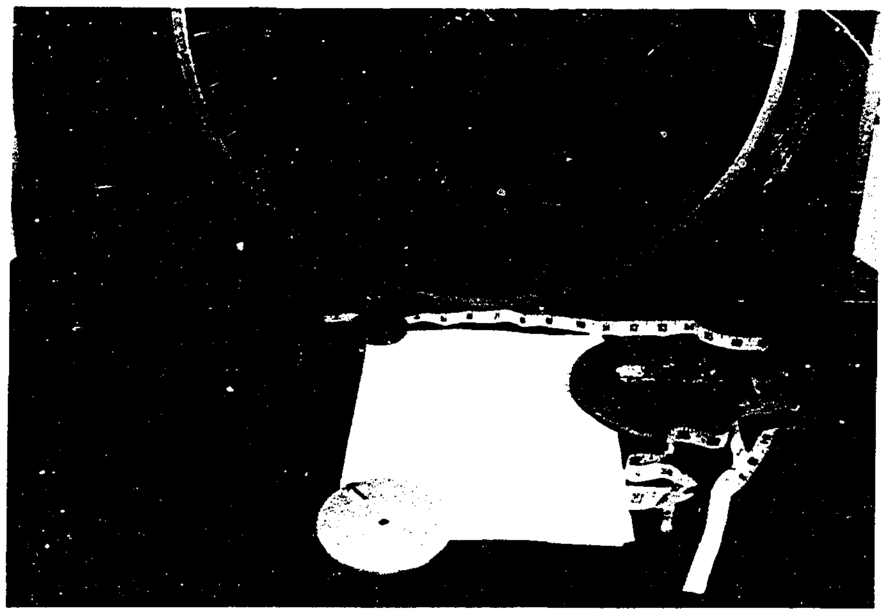


Figure 7. Calculation of pi (π)

The pupils will measure the diameter and circumference of the circular regions, and then divide the measure of circumference by the measure of diameter to get an approximation for π. Keep in mind that this is only an approximation for the number π. The more observations, the better the approximation.

DIAMETER AND CIRCUMFERENCE OF A CIRCLE

In this shoe box you are asked to do some measurement.

First, measure the diameter of each circular wheel. Record this in the table.

Next, measure the circumference. Circumference is the distance around a circle's boundary. To measure this distance, put a pencil in the center of the circular wheel and roll it around exactly once along the tape measure or yardstick. Record the circumference for each circle in the table.

After you have measured the diameter and circumference of the circles, you should fill in column three of the table.

Diameter	Circumference	Circumference ÷ Diameter

All of the numbers in column 3 of your table should be close to 3.1 or 3.2.

Now average* the results of column 3. This will give you an approximation to a very important number in mathematics. Mathematicians call this number Pi (pronounced pie). The symbol is π.

For your experiment

π = ________

*To average a set of numbers you must add them up, then divide by the number of addends. For example, the average of the numbers 6, 9, 5, 8 is $(6 + 9 + 5 + 8) \div 4$, because there are 4 addends.

$28 \div 4 = 7$, so 7 is the average of the 4 numbers.

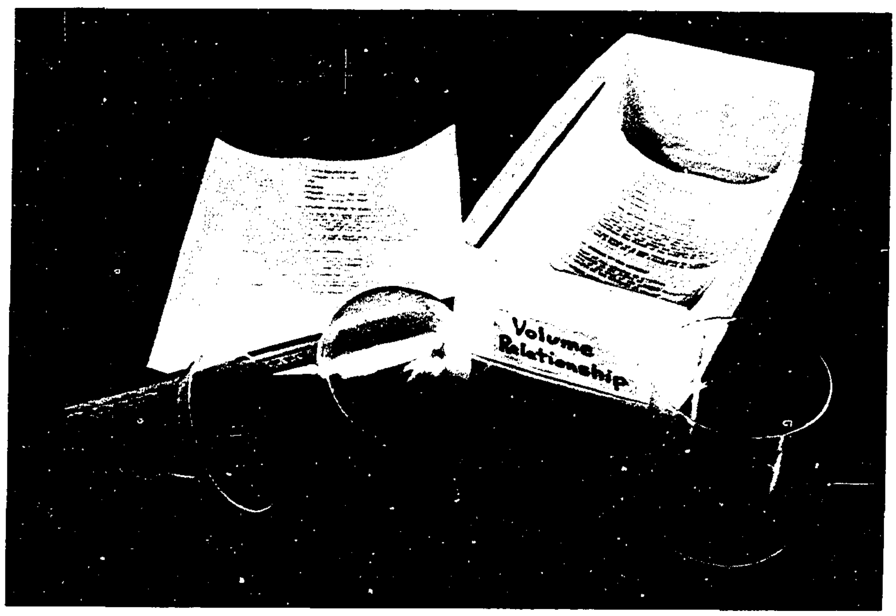


Figure 8. Volume relationship

Pupils will measure the geometric shapes. The most efficient way to determine the volume of the cone and sphere is to fill the cone with water and proceed from there.

The contents of this box are a sphere, a cone, and a cylinder.

A. MEASURE EACH OF THESE SHAPES

Sphere________ in. diameter

Cone __________ in. diameter, ________ in. high

Cylinder ______ in. diameter, ________ in. high

B. The volume of the cylinder is approximately 21 cubic inches.

C. Next you will find the volume of the cone.

Just by looking at the cone and cylinder, how does the volume of the cone compare to the volume of the cylinder?

Volume of cone is ___________ the volume of the cylinder.

less than (<)
more than (>)
equal to (=)

We know the volume of the cylinder is 21 cubic inches. What could you do to find the volume of the cone? Talk it over and think it over. If you can't do it on your own, a hint is given in the answer box.

The volume of the cone is approximately ______ cubic inches.

If the volume of a cylinder is $V = \pi r^2 h$,* the volume of a cone is V = ____________________.

How did you solve the problem?

*$V = \pi r^2 h$ means to multiply π ($\pi = 3.14$) times the radius of the cylinder squared (r x r) then multiply times the height of the cylinder. V = 3.14 x r x r x h

Did you need the hint?

D. Next you will find the volume of the sphere. You will need both the cone and the cylinder to solve this problem.

The volume of the sphere is ___________ the volume of the cylinder.

equal to
greater than
less than

The volume of the sphere is approximately _______ cubic inches.

E. Write a formula for the volume of:

1. a cylinder; V = _______________.
2. a cone; V = ___________________.
3. a sphere; V = _________________.

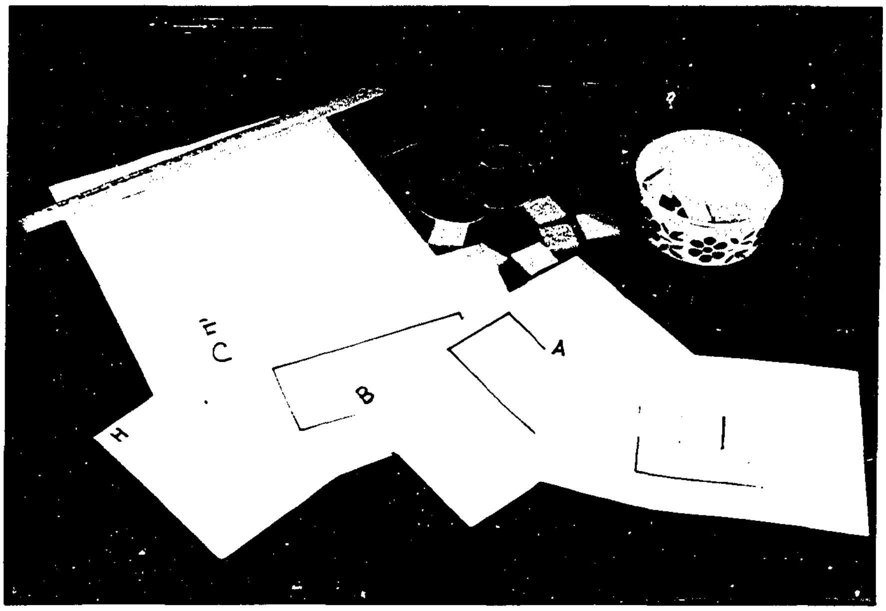


Figure 9. Area of a rectangle

Pupils will generalize the formula for the area of a rectangle. The string will be used to approximate perimeter, and the tile will be used to approximate area.

AREA AND PERIMETER OF A RECTANGLE

This shoe box contains a box of squares (square units), a ruler, and drawings of rectangles.

Use the square tiles to measure the area.

Use the ruler or the string and ruler to measure the perimeter.

Look for a rule for computing the area and perimeter of a rectangle.

	LENGTH	WIDTH	AREA HOW MANY TILES?	PERIMETER HOW FAR AROUND?
Rectangle A				
Rectangle B				
Rectangle C				
Rectangle D				
Rectangle E				
Rectangle F				
Rectangle G				
Rectangle H				
Rectangle I				

Complete the following mathematical sentences:

Area : A = ________________

Perimeter : P = ________________

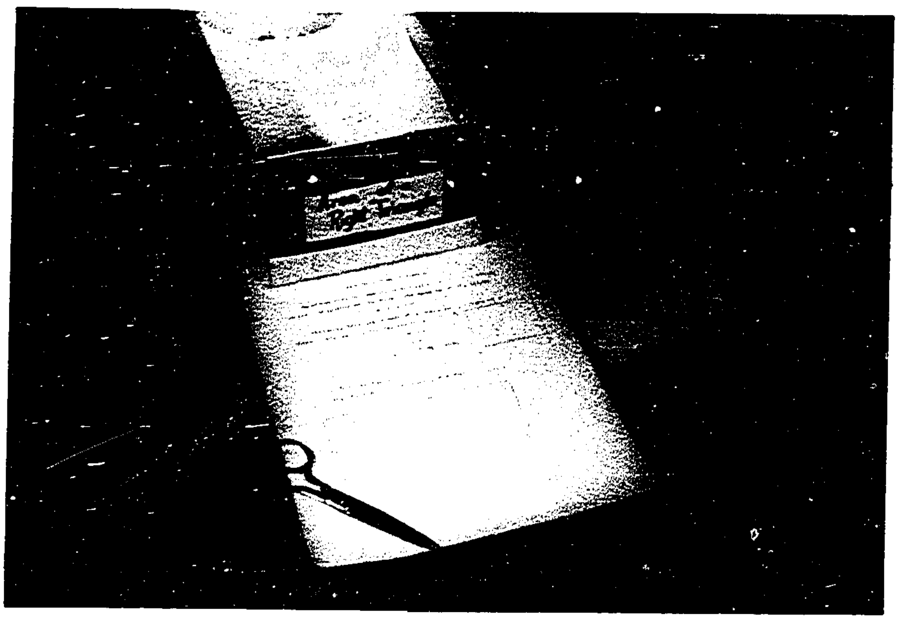


Figure 10. Area of a right triangle

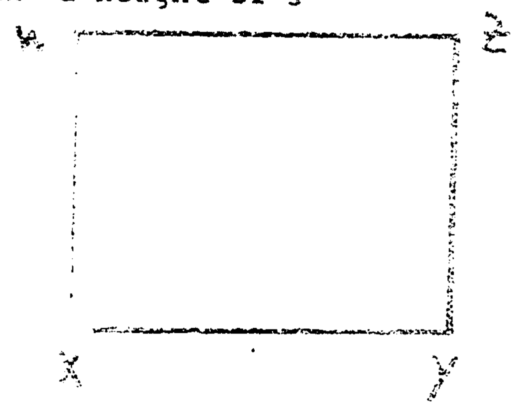
The pupils will cut the rectangle in half and observe that two congruent right triangles are formed. Thus, the area of the right triangle must be ½ the area of a parallelogram ($A = \frac{1}{2}b \times h$).

AREA OF A RIGHT TRIANGLE

Remember that the area of a rectangle is found by multiplying the measure of the base x the measure of the height.

$$A = b \times h$$

Rectangle WXYZ has a base of 4 units and a height of 3 units. The area of the rectangle is



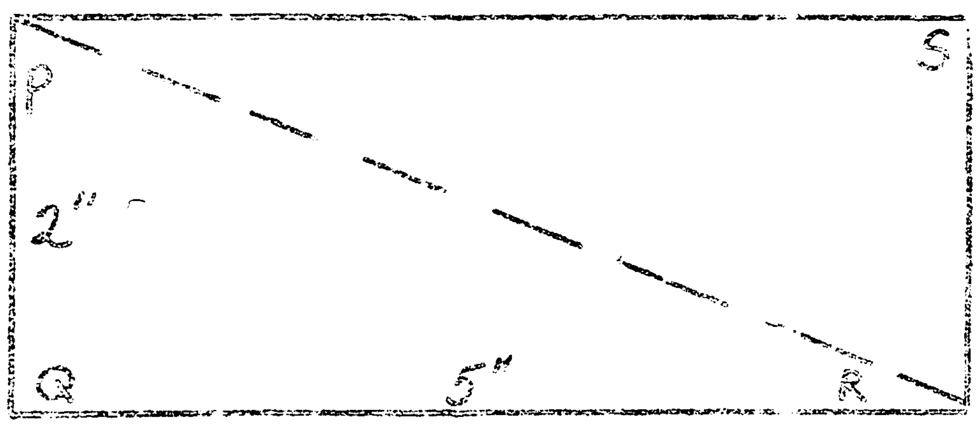
$$A = b \times h$$

$$A = 4 \times 3$$

$$A = 12 \text{ sq. units}$$

Look at rectangle PQRS.

What is the area of rectangle PQRS? ________



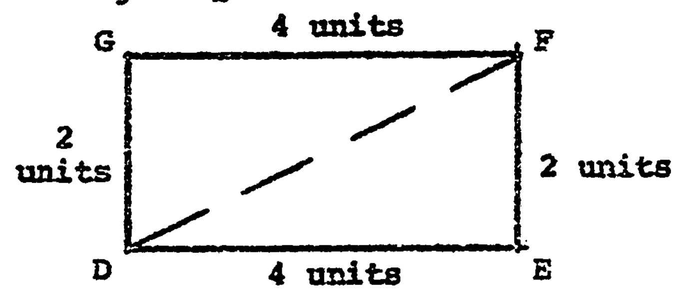
Now take the scissors and cut out rectangle PQRS.

You should have said the area of PQRS is 10 square inches.

Now that you have cut out the rectangle PQRS, cut the rectangle on the diagonal PR.

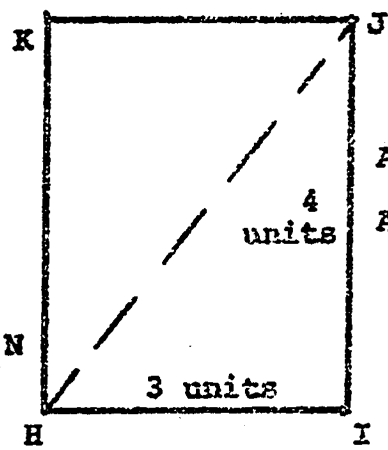
You now have 2 right triangles. They each have a right (90^{o}) angle. Are these 2 triangles equal in area? __________

How will the area of triangle PQR compare with the area of rectangle PQRS? Write the answer you think is correct: The area of triangle PQR is ________________________ the area of rectangle PQRS.
(twice, one-half, equal to)



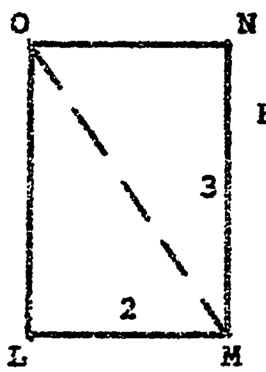
Area of rectangle DEFG = ______

Area of triangle DEF = ______



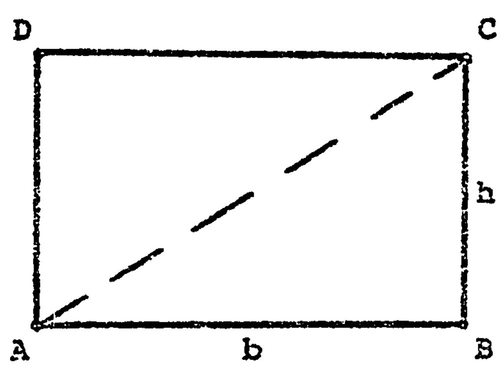
Area of rectangle HIJK = ______

Area of triangle HIJ = ______



Area of rectangle LMNO = ______

Area of triangle MNO = ______



Area of rectangle ABCD = ______

Area of triangle ABC = ______

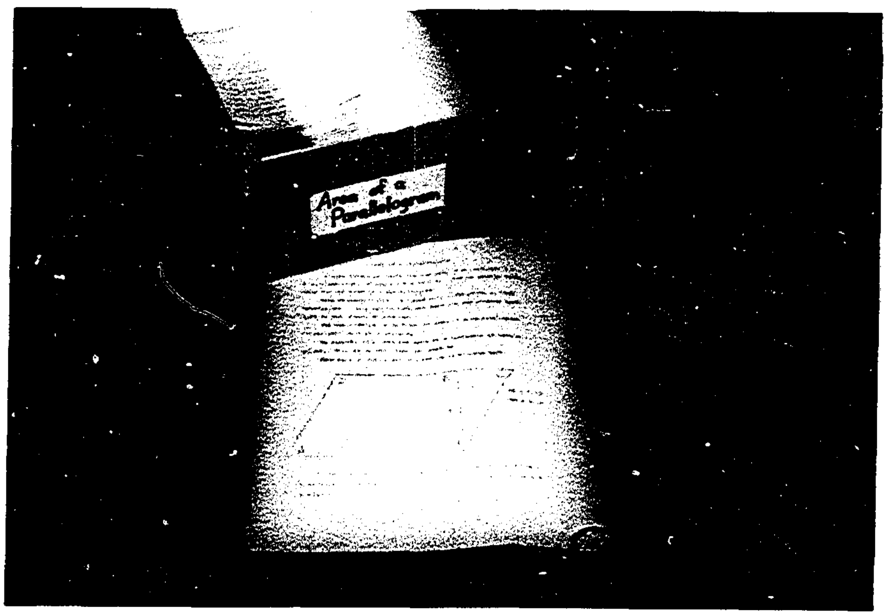


Figure 11. Area of a parallelogram

The pupils will cut a parallelogram into two pieces in such a way that the two pieces can be rearranged to form a rectangle. The area, base, and height of the parallelogram will be equal respectively to the area, base, and height of the rectangle. Consequently the area of the parallelogram is also $A = b \times h$.

AREA OF A PARALLELOGRAM

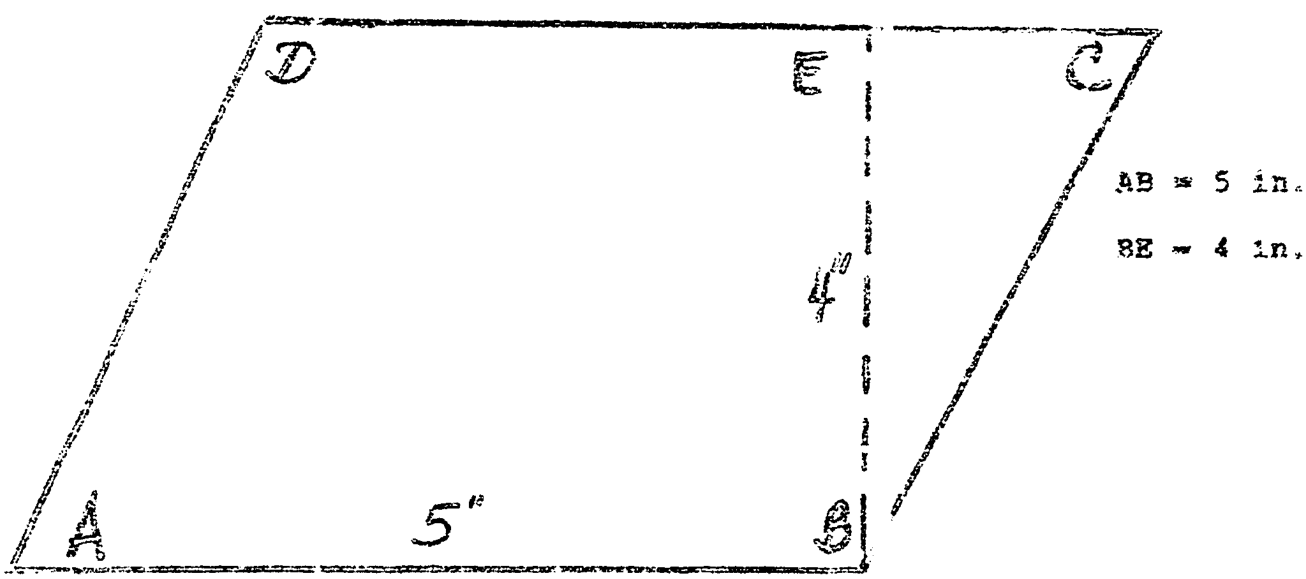
We have developed rules for the area of a rectangle (A = b x h) and right triangle (A = ½b x h). Now we need to find a rule for the area of a parallelogram.

When mathematicians are faced with a new problem, they usually ask themselves if they can make the new problem look like an old, familiar problem.

The new problem is to find the area of a parallelogram. Figure ABCD is a parallelogram.

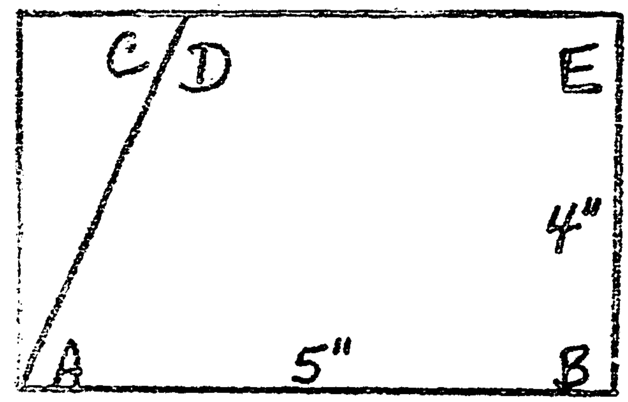
We already know that the area of a rectangle is base x height. This is an old, familiar problem.

Take your scissors and cut out parallelogram ABCD.



If you now cut along dotted line BE, you can take the triangular piece and rearrange the pieces so that you have a rectangle. Do it.

One way to rearrange the pieces is



and this is our familiar shape, the rectangle. The area of the rectangle with base $\overline{AB}$ and height $\overline{BE}$ is:

A = b x h
A = 5 x 4
A = 20 square inches

Now put the triangular piece back to its original position and you have the parallelogram ABCD. The height of the rectangle is 4 inches. The height of the parallelogram is <u>also</u> 4 inches. The <u>height</u> of a parallelogram and the rectangle you make from it will always be <u>equal</u>.

1. Will the area of parallelogram ABCD equal the area of the rectangle at the top of this page? ____________

On the next page there are 2 parallelograms. Cut out parallelogram GHIJ and make it into a rectangle.

2. If the base of the rectangle is 6" and the height is 3" what is the area? ____________

3. What is the area of the parallelogram GHIJ? ____________

Cut out the parallelogram KLMN and convert it into a rectangle.

4. Do the rectangle and the parallelogram have the same area? ______

5. If the area of the rectangle is 18, what is the area of the parallelogram? ____________

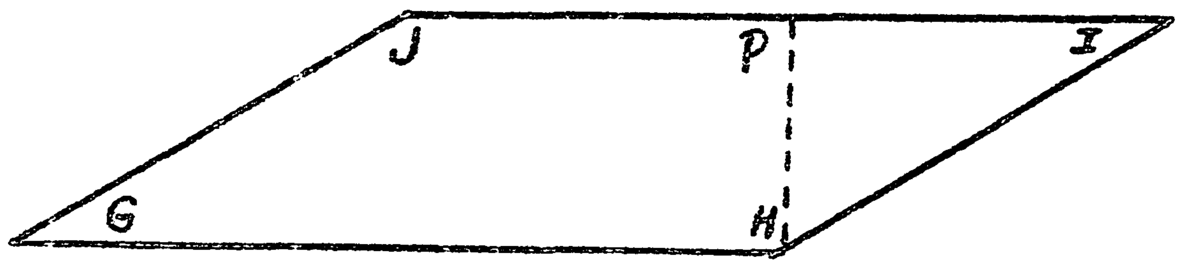
6. The area of a rectangle is b x h. Since any parallelogram can be rearranged into a rectangle, the area of a parallelogram is:

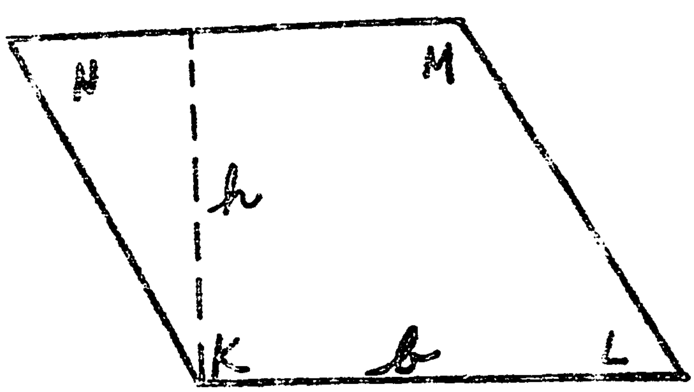
A = ________________

The height of parallelogram ABCD is $\overline{BE}$. Does $\overline{HI}$ represent the height of parallelogram GHIJ? ____________

Your answer should be no. The height of parallelogram GHIJ is $\overline{HP}$. Notice that $\overline{HP}$ is also the height of the rectangle.

The height of a geometric shape must be perpendicular to (form right angles with) the base.





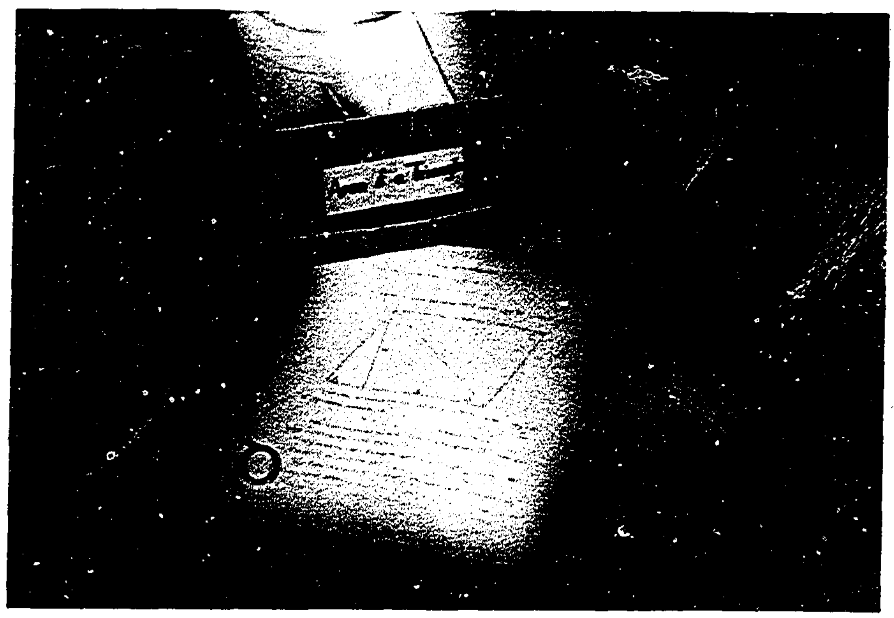


Figure 12. Area of a triangle

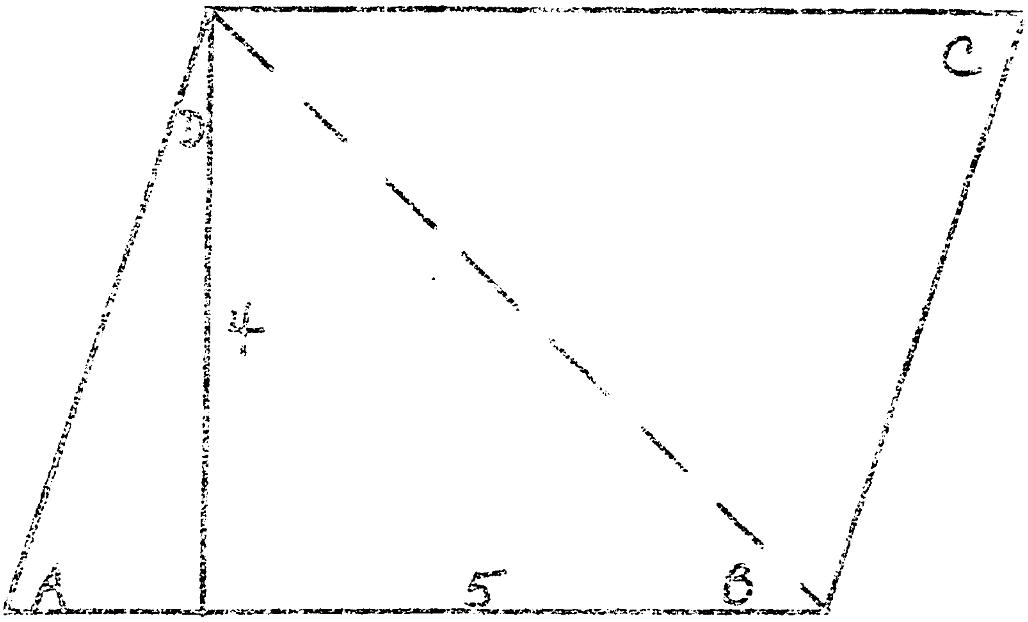
The pupils will cut the parallelogram into two congruent triangles. Since the area of the parallelogram is $A = b \times h$, the area of the triangle must be $A = \frac{1}{2}b \times h$.

AREA OF A TRIANGLE

The area of a parallelogram is

$$A = b \times h$$

For example, parallelogram ABCD has a base of 5 and a height of 4.



Use your scissors to cut out this parallelogram. The area of the parallelogram is

A = ______________ square inches.

Use your scissors to cut along the diagonal $\overleftrightarrow{DB}$. Now you have 2 triangles. Are they equal? __________

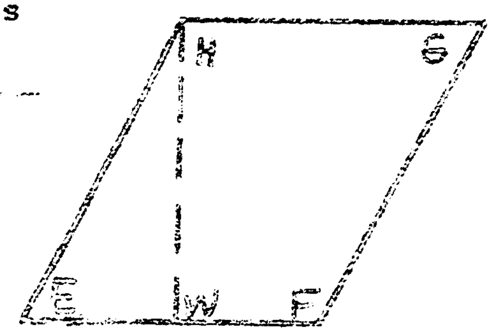
The area of parallelogram ABCD was 20 square units. What is the area of triangle ABD? __________

How does the area of triangle ABD compare to the area of parallelogram ABCD? ____

____

You may cut out the parallelograms if you wish.

The area of parallelogram EFGH is 24. What is the area of triangle EFH? ____________

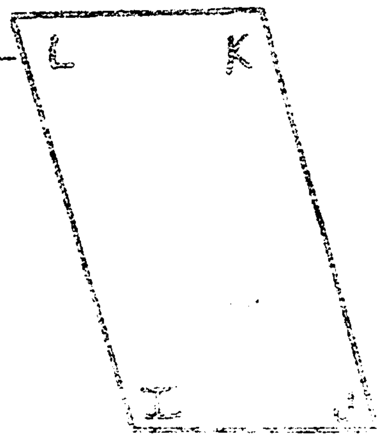


base = 6 = EF
height = 4 = HW

The area of parallelogram IJKL is 18.

What is the area of triangle IJK? ____________

base = 3
height = 6



The area of parallelogram RSTU is b × h.

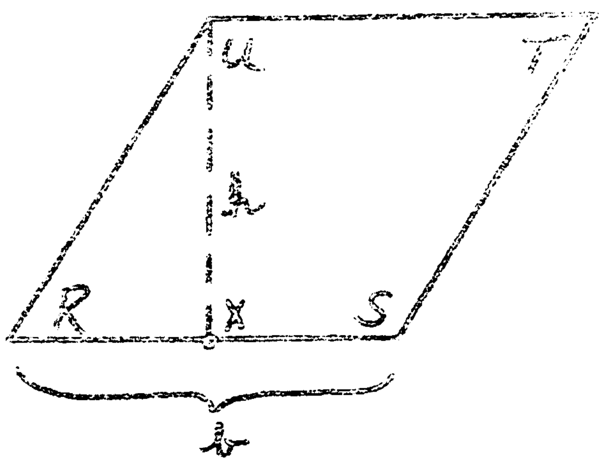
RS = b
UX = h

What is the area of triangle RST?

A = ____________

What is the area of triangle RSU?

A = ____________



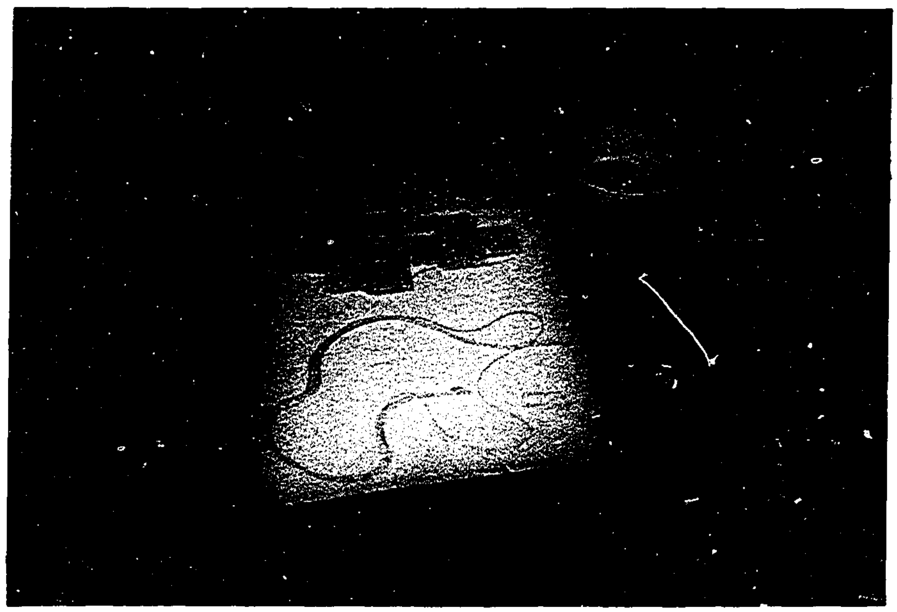
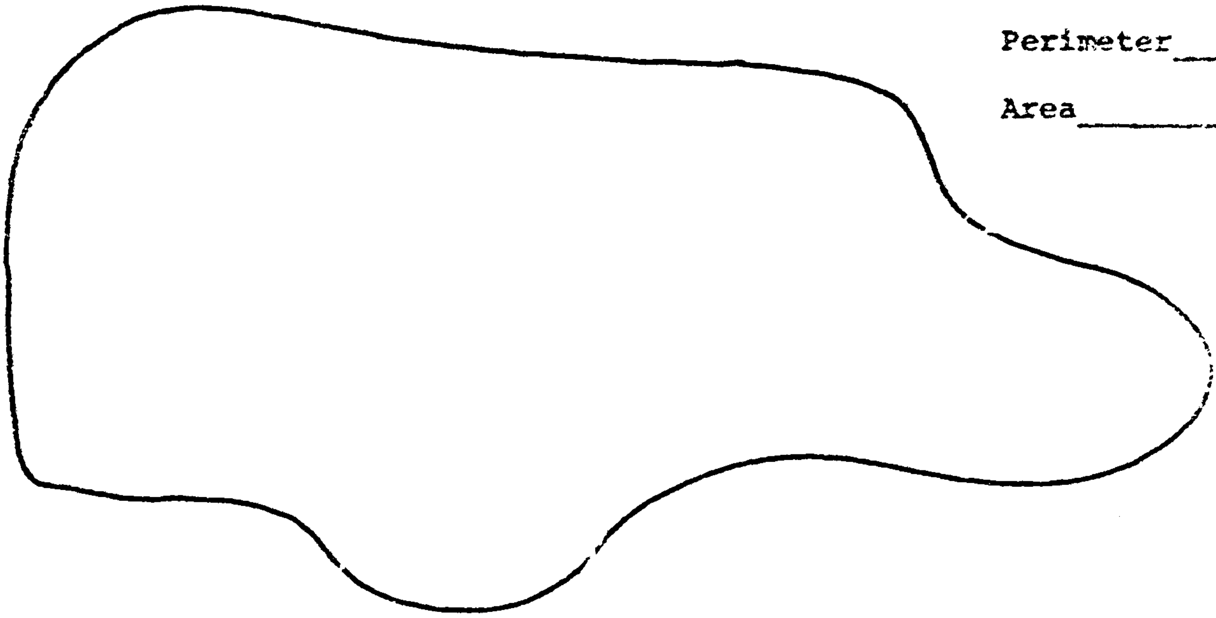


Figure 13. Area and perimeter

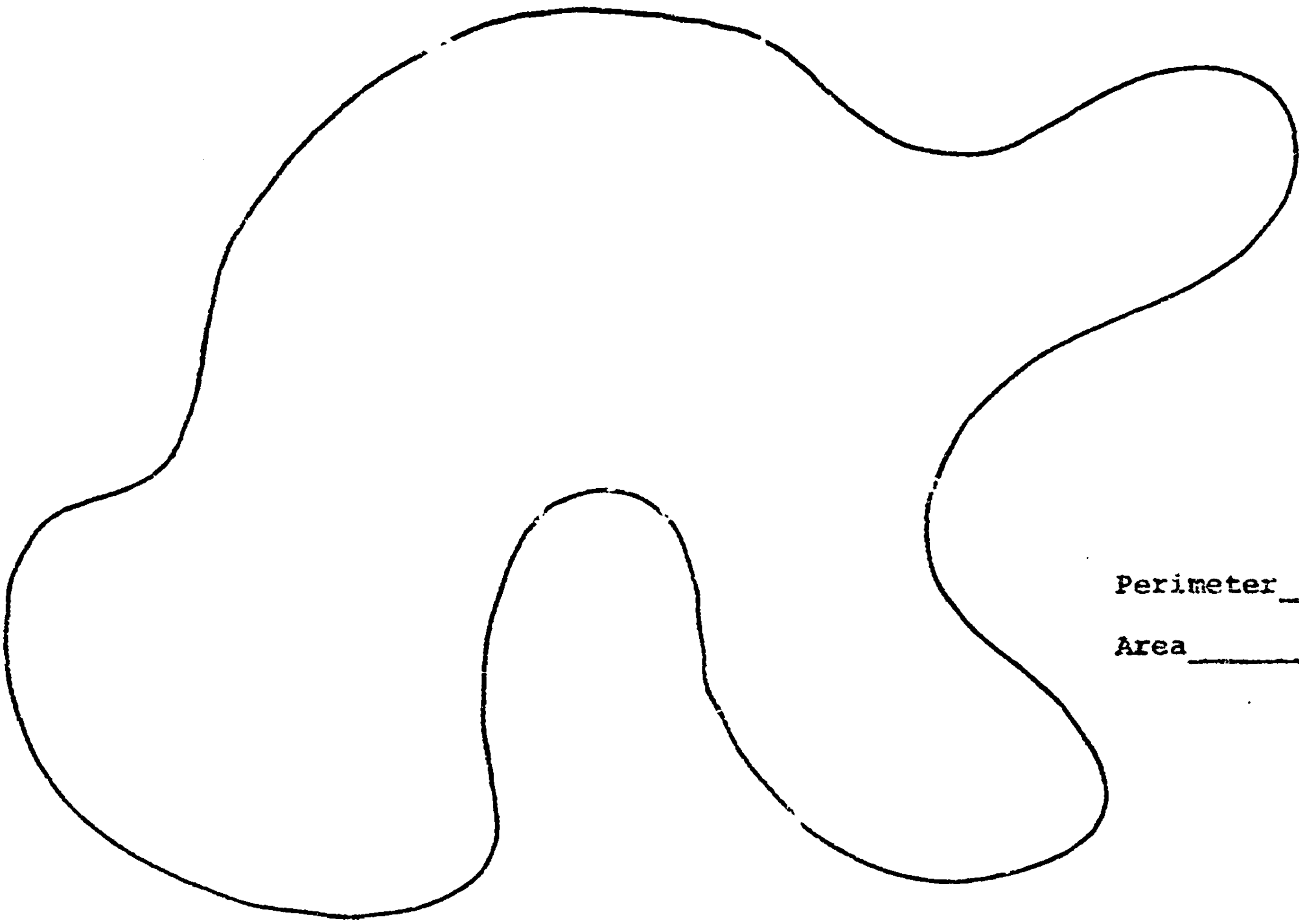
Pupils will "tile" the interior of the shape to get an approximation of the area. Encourage an approximation technique which will give the closest approximation to the "true area," i.e. position tile so that errors will tend to cancel or offset one another. The string will be used to measure the perimeter.

Find the perimeter and area for the following simple closed curves. Use the square tiles to get the area. Use the ruler and/or the string to get the perimeter.



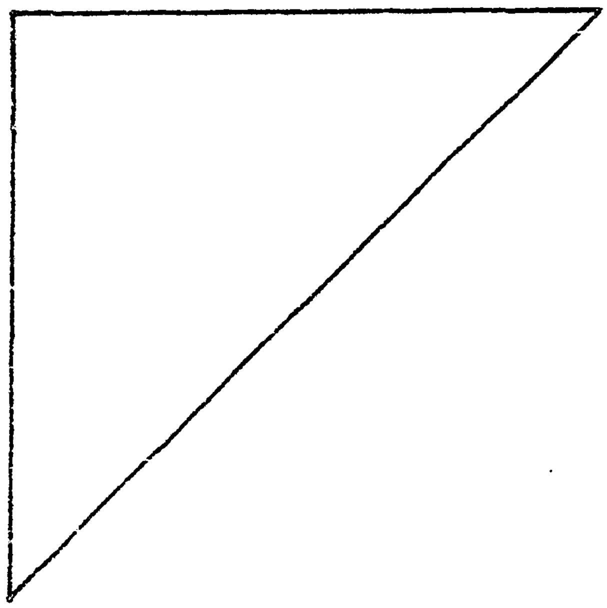
Perimeter________

Area________



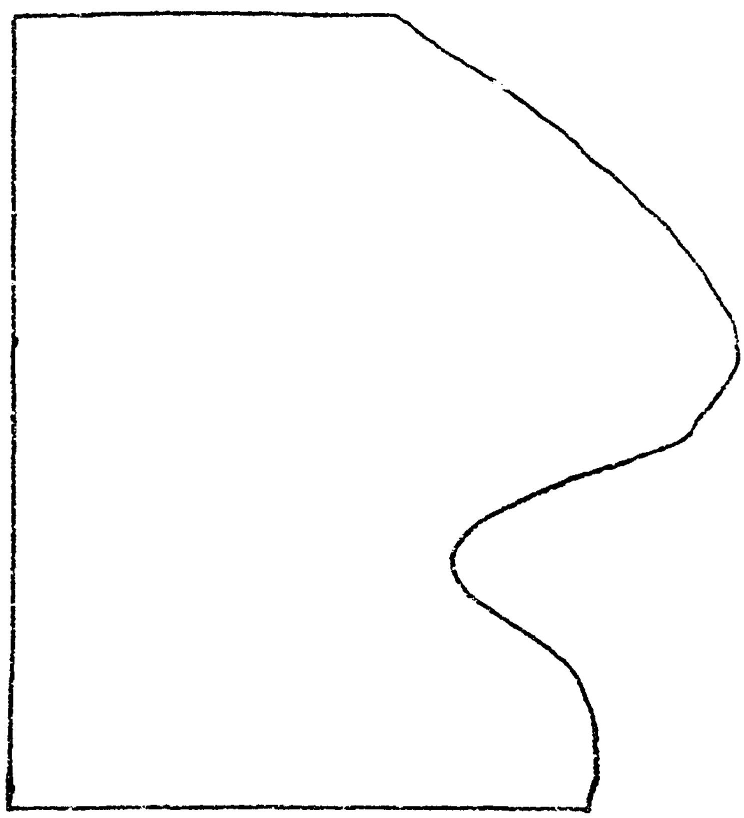
Perimeter________

Area________



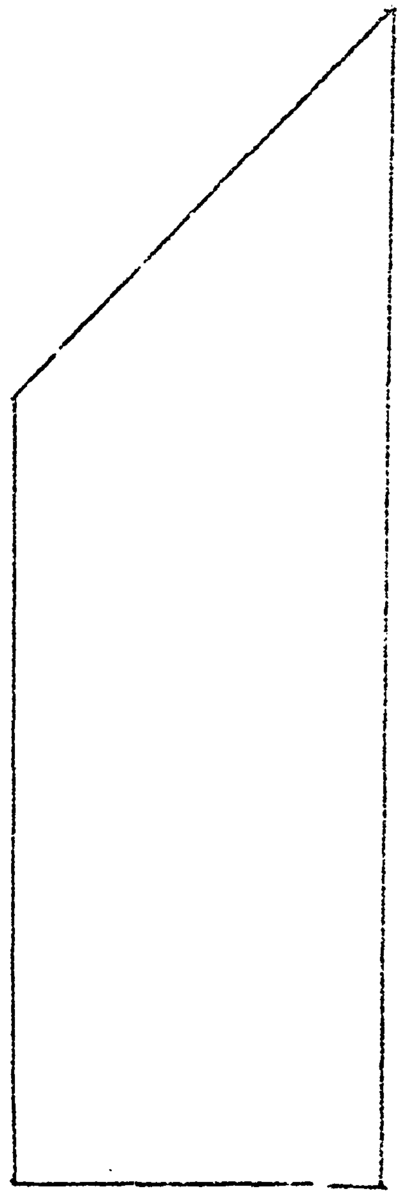
Perimeter________

Area________



Perimeter________

Area________



Perimeter________

Area________

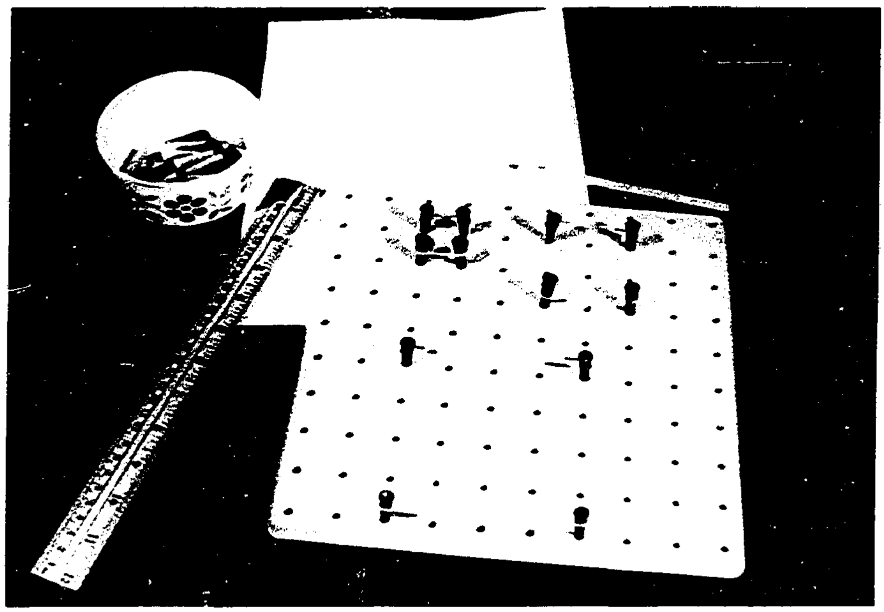


Figure 14. Side-area relationships

Pupils will form squares using rubber bands. The first square should be one unit on a side. Next, the pupil is asked to form a square two units on a side. The area for this new square is four times as big as the area of the first square. The pupil will continue to increase the length of the side and record the corresponding increase in the area. The generalization of this pattern is fairly difficult for sixth grade pupils.

SIDE - AREA RELATIONSHIP

Get a geoboard and some rubber bands.

With one rubber band make a square 1 unit on a side. Take another rubber band and make a square 2 units on a side. With another band make a square 4 units on a side. Finally, make a square 8 units on a side.

Length of Side	Area
1	
2	
4	
8	

When you double the side of a square, the area is ________ times as big.

Example:	Side = 1	Area = ______
	Side = 2	Area = ______
Another example:	Side = 2	Area = ______
	Side = 4	Area = ______
Another example:	Side = 4	Area = ______
	Side = 8	Area = ______

Complete this table using the geoboard and rubber bands.

Length of Side of Square	Area of Square
1	
3	
9	

If we make the side of a square three times as long, we make the area ________ as big.

Example: Side = 1 Area = ______

Side = 3 Area = ______

Another example: Side = 3 Area = ______

Side = 9 Area = ______

If we multiply the side of a square by 4, how many times bigger is the area? ________________________________

If we multiply the side of a square by 7, how many times bigger is the area? ________________________________

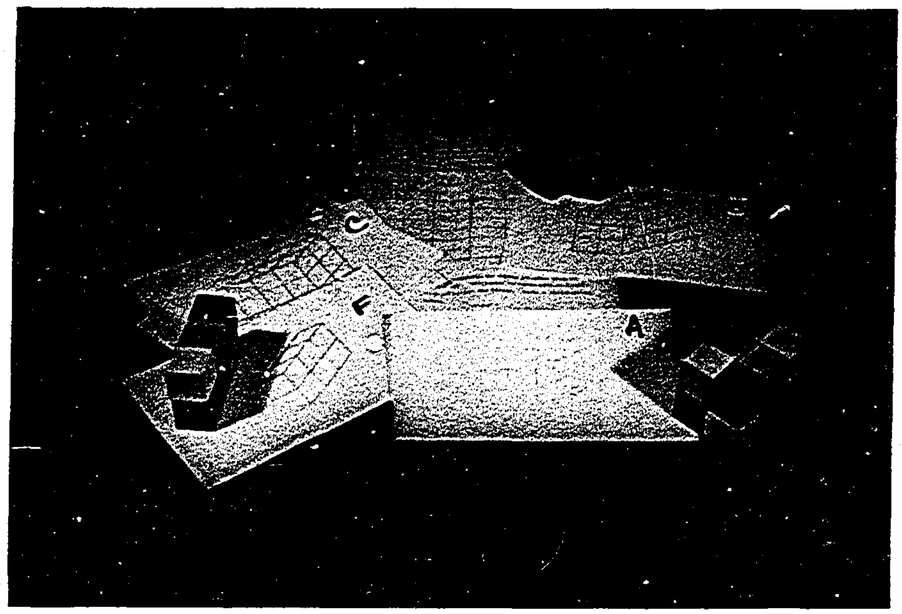


Figure 15. Rectangular prisms

The pupil will use one inch cubes to build the shapes which are pictured on the cards. Next, the pupil will observe that he can predict the number of cubes needed to build the prism by finding the product of the three dimensions. Finally he will write the general formula for the volume of a rectangular prism.

This shoebox contains a set of cubes (cubic units) and pictures of some rectangular prisms.

Use the pictures to help you build each rectangular prism. [illegible]fter you have built each rectangular prism, write the number for [illegible] length, width and height. In the last column record how many [illegible]bes it takes to build the rectangular prism.

[illegible]cture	Width	Length	Height	How many cubes used to build the prism
A	2	3		
B			3	
C	4			
D				
E				
F				
G				

Look for a rule for computing the number of cubes used to build each rectangular prism. The number of cubes used to build each rectangular prism is called the volume. [illegible]rite a mathematical sentence for the volume.

Volume = ______________

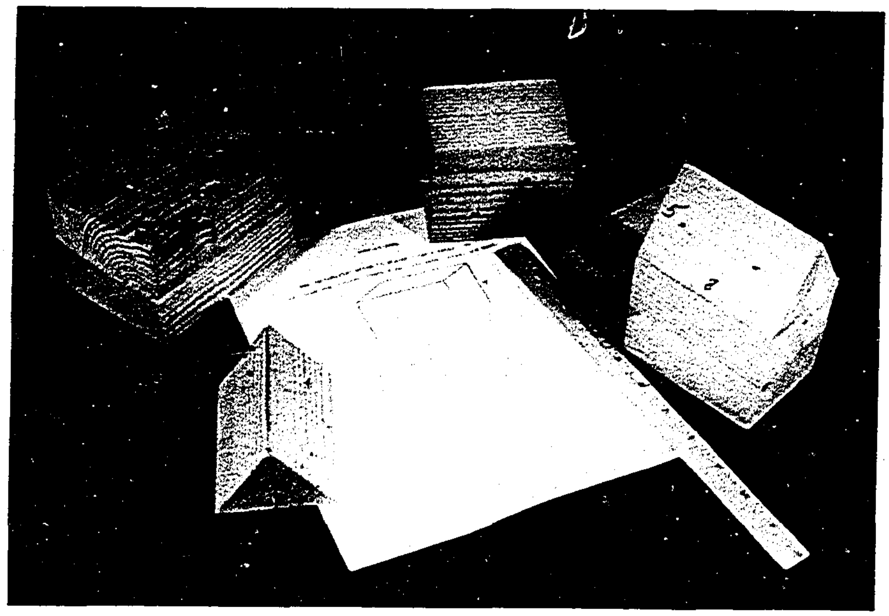
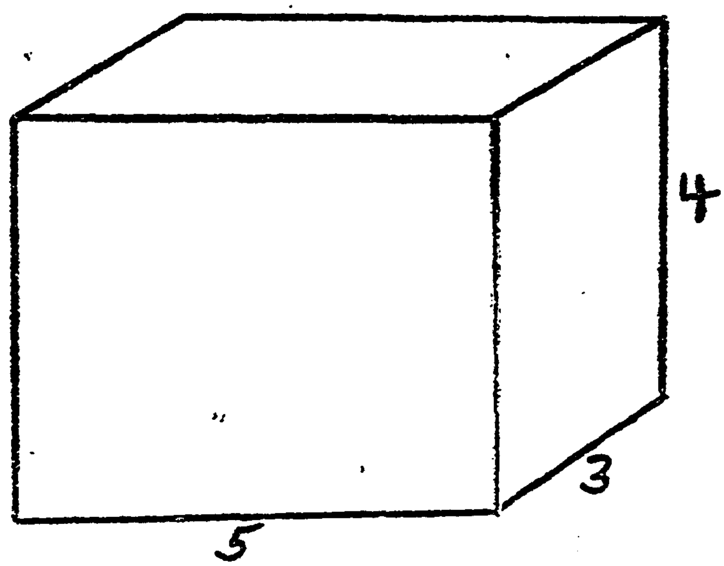


Figure 16. Surface area

The pupil will measure the lengths of the edges of the shapes provided and calculate the total surface area of the wooden shapes.

SURFACE AREA

The rectangular prism in Figure 1 has 6 faces. We will call them top, bottom, right side, left side, front, and back.



Calculate the area of each <u>face</u>:

FACE	LENGTH	WIDTH	AREA
right side	3	4	12
front	5	4	
top			
bottom			
back			
left side			

TOTAL SURFACE AREA =

Find the surface area for Figure 2.

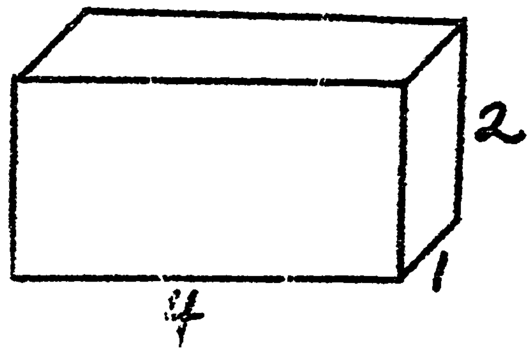


Figure 2.

FACE	LENGTH	WIDTH	AREA
right side			
front			
top			
bottom			
back			
left side			

TOTAL =

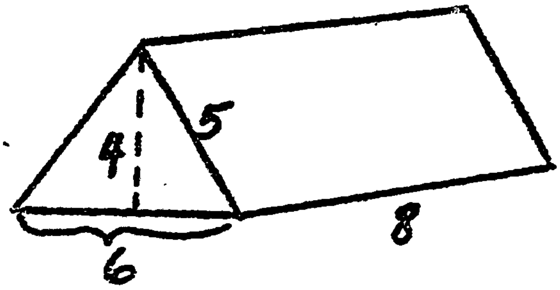


Figure 3.

FACE	BASE	HEIGHT	AREA
triangle front	6	4	
right side		5	
triangle back			
left side			
bottom			

TOTAL =

Find the total surface area for Figure 4.

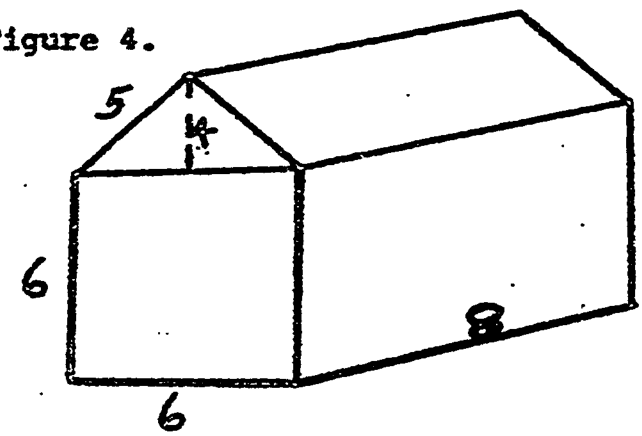


Figure 4.

What percent of this total surface area is floor space? ___________

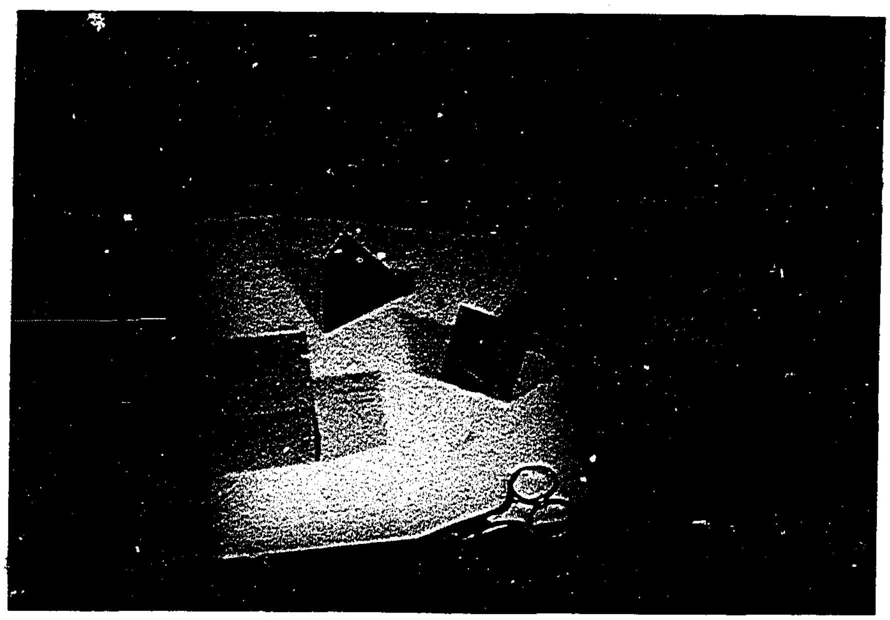


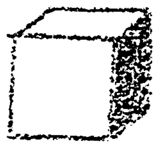
Figure 17. Construction of polyhedra

The pupil will trace the flat lay-out of the regular polyhedra on construction paper and cut it out. Next, he will use tape to form the three dimensional polyhedra. Neatness and accuracy are important for this construction. More difficult constructions are given in the enclosed book.

In this box you will find five patterns. You may use them to build models of the five regular polyhedra which are sketched below. Cut them out. You can build the shape using this paper, or you may trace onto construction paper and make the polyhedra out of construction paper. Use scotched tape to fasten the edges together. Cut only on the solid lines.

Tetrahedron

4 Faces

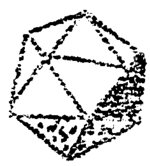


Hexahedron or Cube

6 Faces

Octahedron

8 Faces

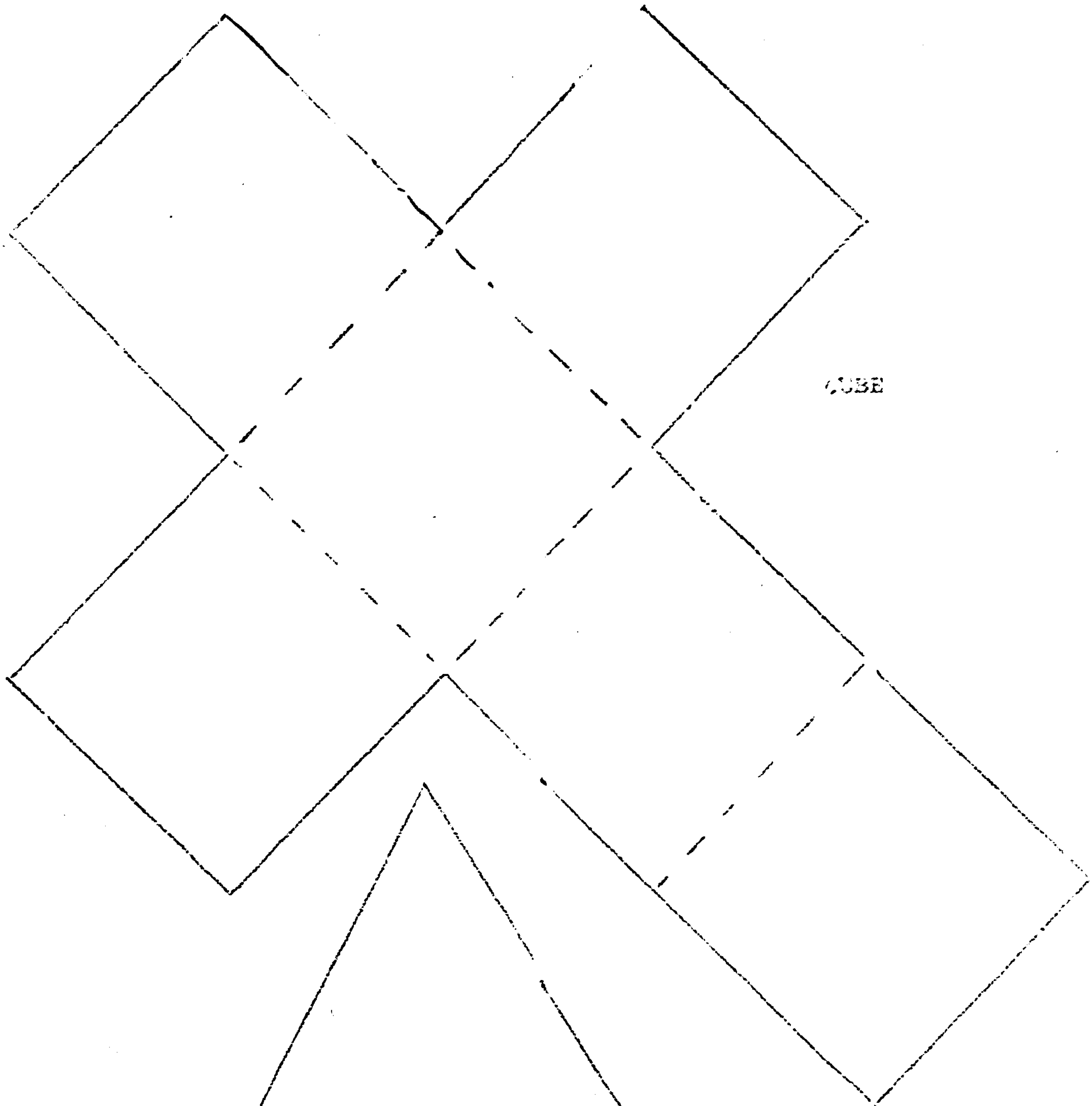


Icosahedron

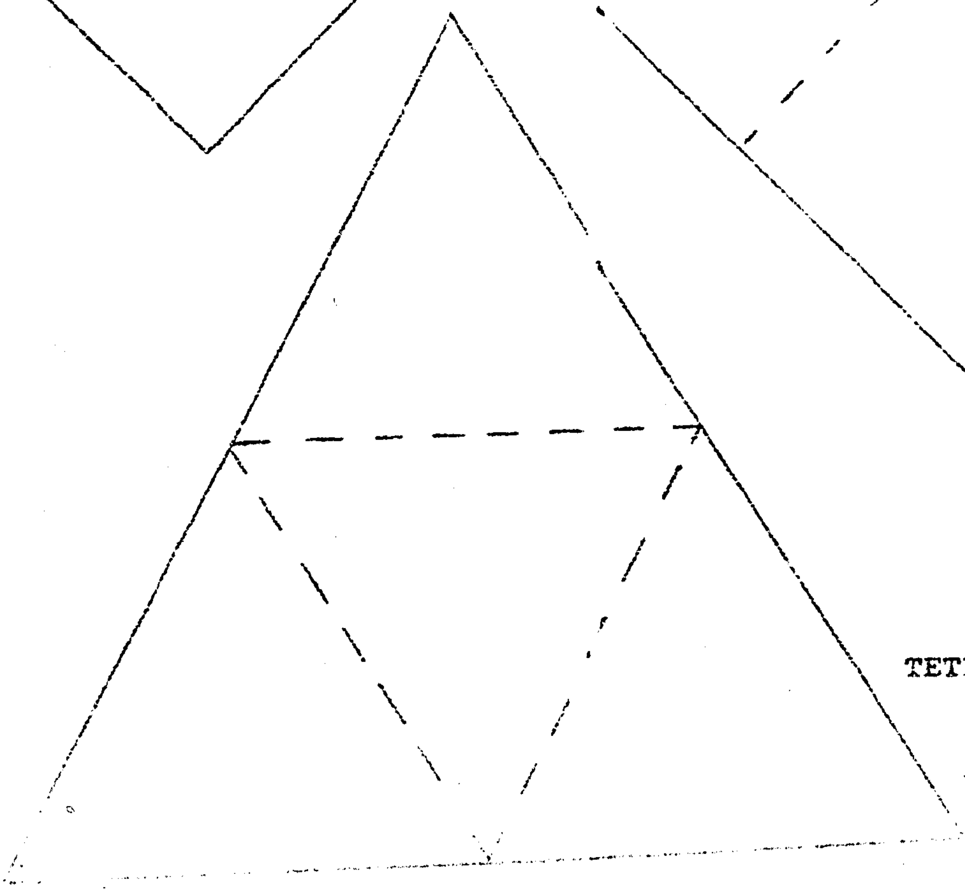
20 Faces

Dodecahedron

12 Faces

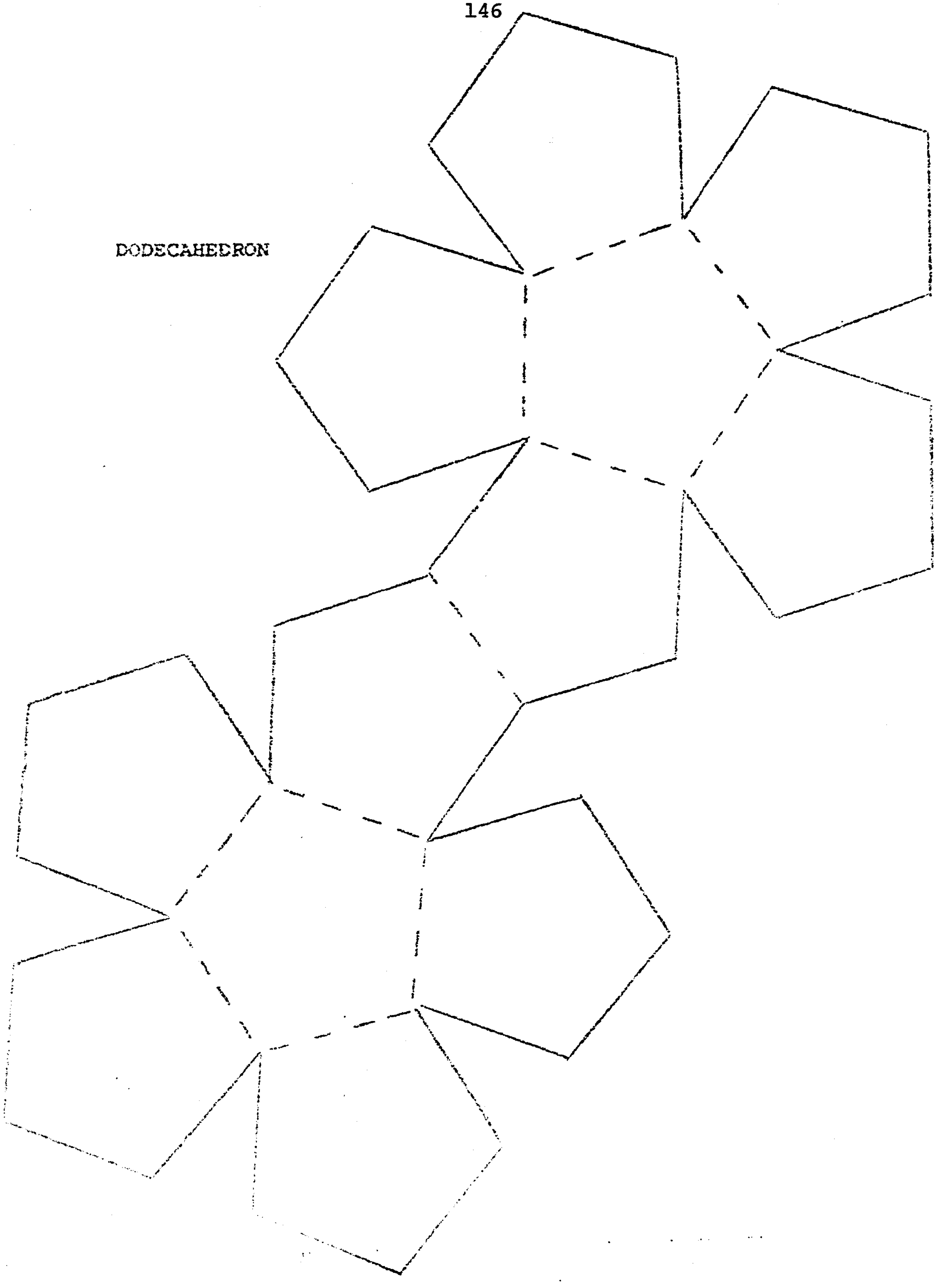


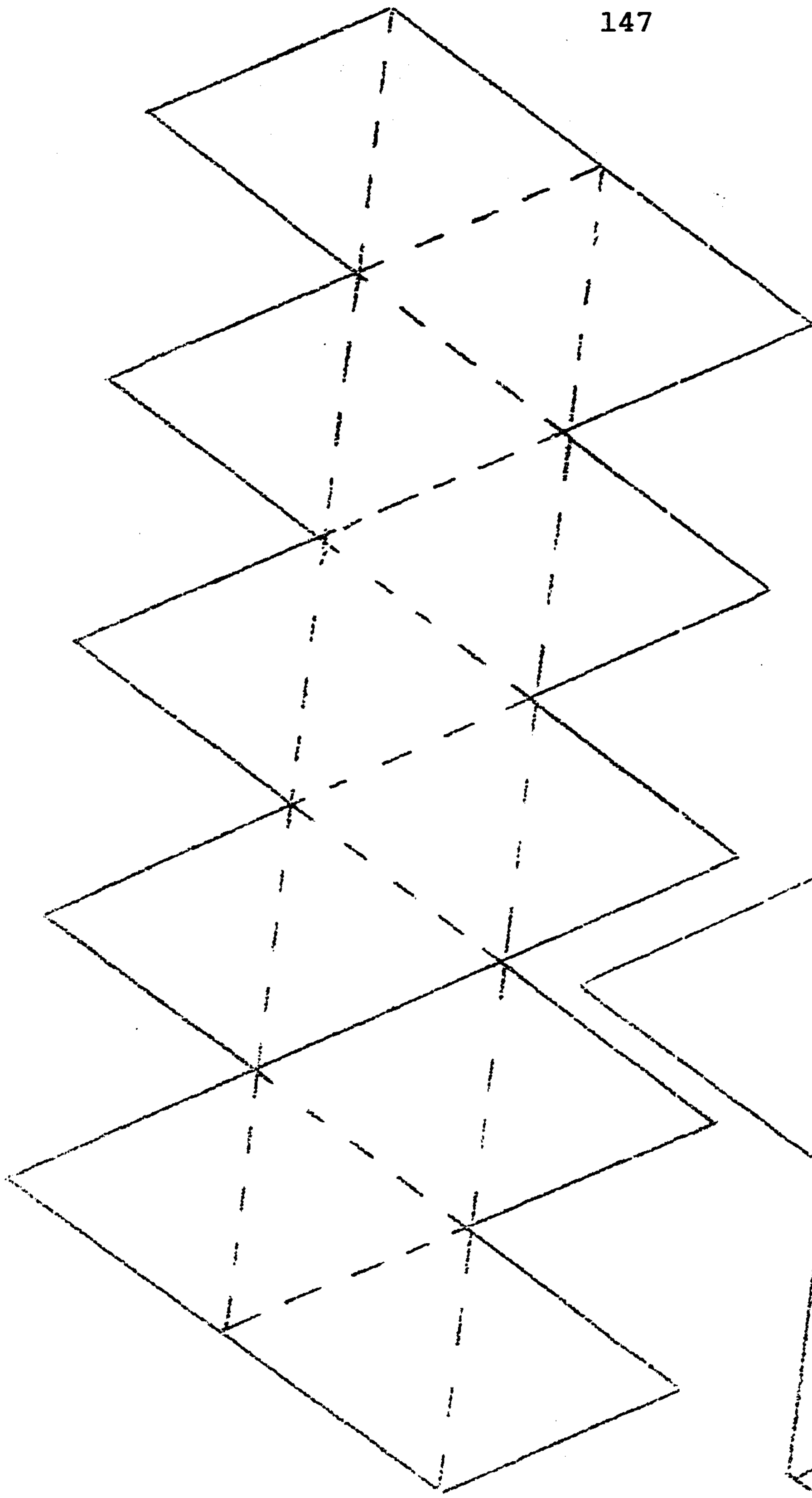
CUBE

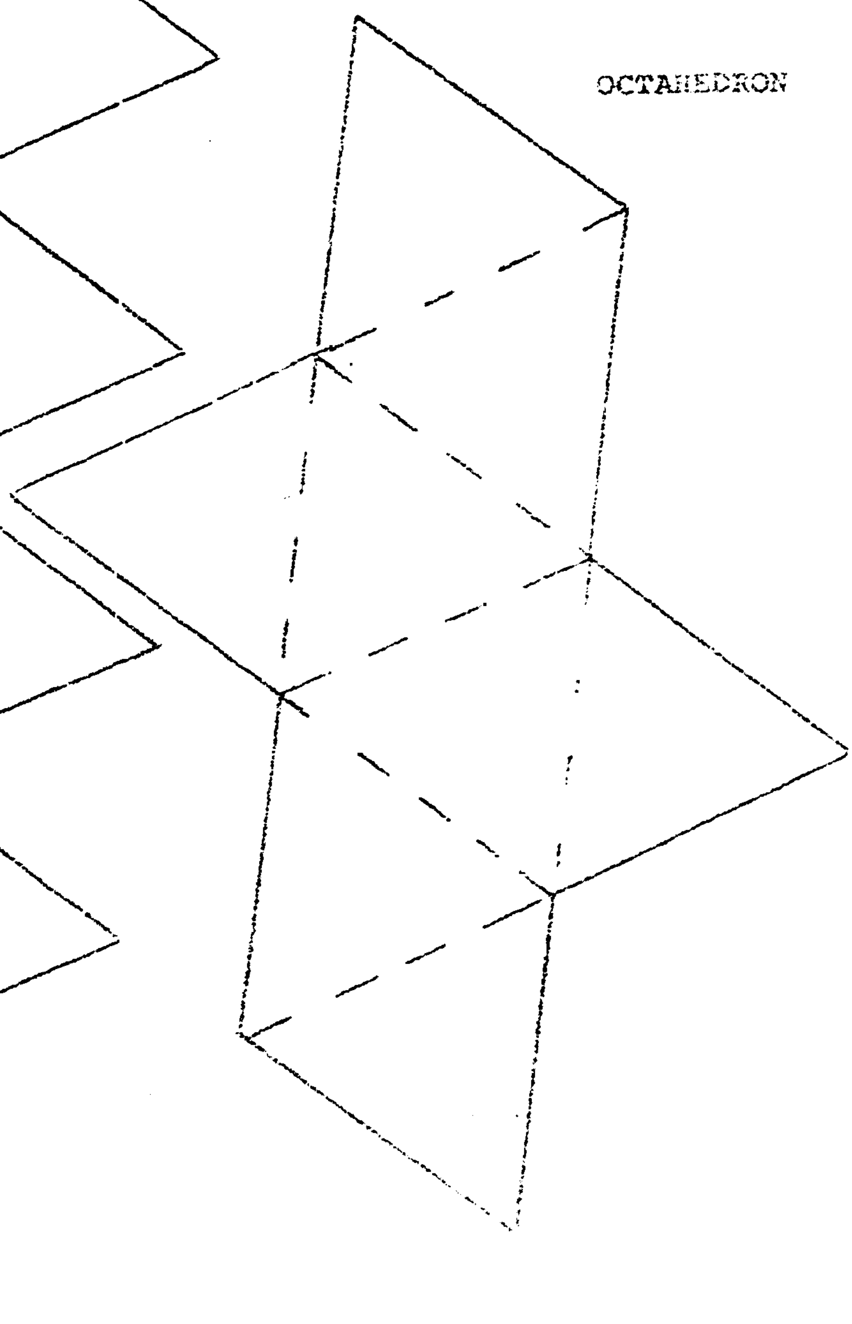


TETRAHEDRON

DODECAHEDRON







OCTAHEDRON

ICOSAHEDRON

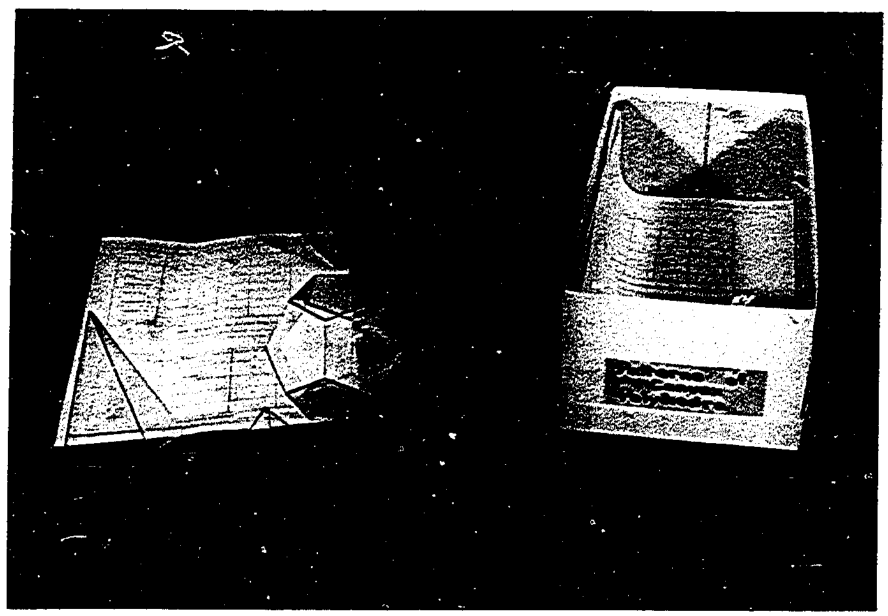
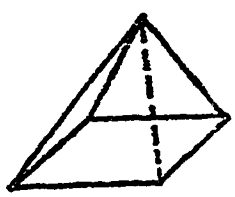


Figure 18. Euler's formula

This is an exercise in counting the number of vertices (V), edges (E), and faces (F) of regular polyhedra. The pupil will observe that V + F - E = 2 for all regular polyhedra. This is called Euler's Formula.

If you have constructed the models of the regular polyhedra, use them to complete this worksheet. If not, use the models which are provided. Fill in the blanks:

Geometric Shape	Number of vertices V	Number of faces F	Number of edges E	V + F - E
Cube	8	6	12	2
Tetrahedron	4			
Octahedron		8		
Dodecahedron			30	
Icosahedron		20		
Square Pyramid (See sketch) 	5			

What column is always the same? Write the mathematical sentence which states this as a rule. ______

This rule is called Euler's Formula. Euler was a very famous mathematician and he discovered this formula in about 1750.

Complete this table:

Figure	Number of vertices V	Number of regions R	Number of segments S	V + R - S
	4	2 (inside & outside)	4	2
	4			
			8	
				

What conclusions can you make about your results? ______

______

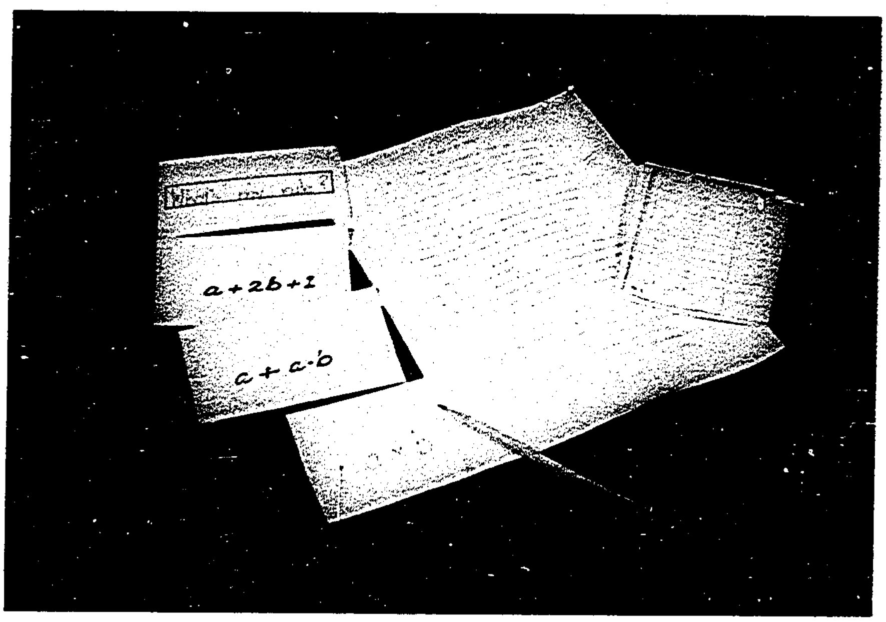


Figure 19. What's my rule?

This is an optional unit which provides pupils with inductive experiences in a game format. It is best to play this game with two players, but four or five could participate. One deck of rules is fairly easy; the other deck is more difficult.

WHAT'S MY RULE?

What's my rule is an arithmetic card game. In the shoe box you will find a score sheet and 2 decks of cards. One deck has yellow dots on them; these are the more difficult ones and should be tried only after you get to know the game.

Take the easier deck and look at the cards. You will notice things like $a + b + 1$ on one card, $2a + b$ on another, and $2(a + b)$ on still another.

The game consists of trying to guess your opponent's rule.

Start the game with 2 players and each one draws a card. Be careful not to let your opponent see your rule.

Take a score sheet and at the top write the rule that is on your card. Now your job is to guess your opponent's rule.

Suppose Mary has the rule $2a + b$ on her card. Bill has the rule $a + b + 1$ on his card. To start the game, Mary says any ordered pair of numbers to Bill. For example, Mary says "2, 5". Now Bill must tell Mary what number his rule would assign to 2, 5. Bill would say "8" because $a + b + 1$, when $a = 2$ and $b = 5$ gives $2 + 5 + 1 = 8$. Mary records this on line one of her score sheet.

Now Bill would give a number pair to Mary. Suppose Bill says "3, 1". Mary would look at her rule, $2a + b$ and say "7", because $2 \times a + b$ when $a = 3$ and $b = 1$ gives $2 \times 3 + 1 = 7$. Bill would record $(3, 1) \rightarrow 7$ on the first line of his score sheet.

Now Mary gives another number pair and so on <u>until</u> one player tries to guess the other player's rule. If you guess the rule after 2 number pairs, you get 90 points. If you guess the rule after 5 number pairs, you get 75 points.

If you try to guess the rule and guess wrong you subtract 20 points from your score.

You can play the game to 250 to decide the winner.

After you get good, try the more difficult deck.

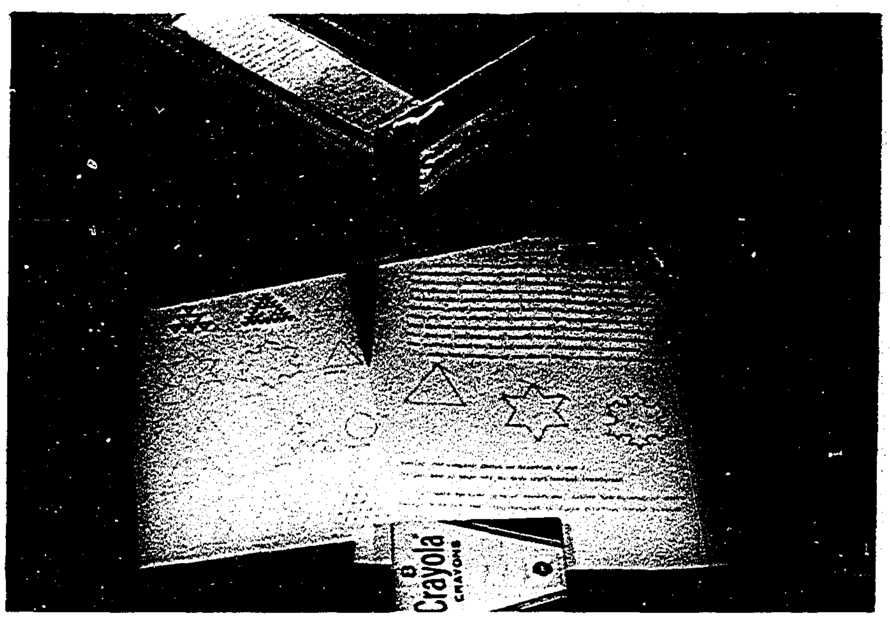


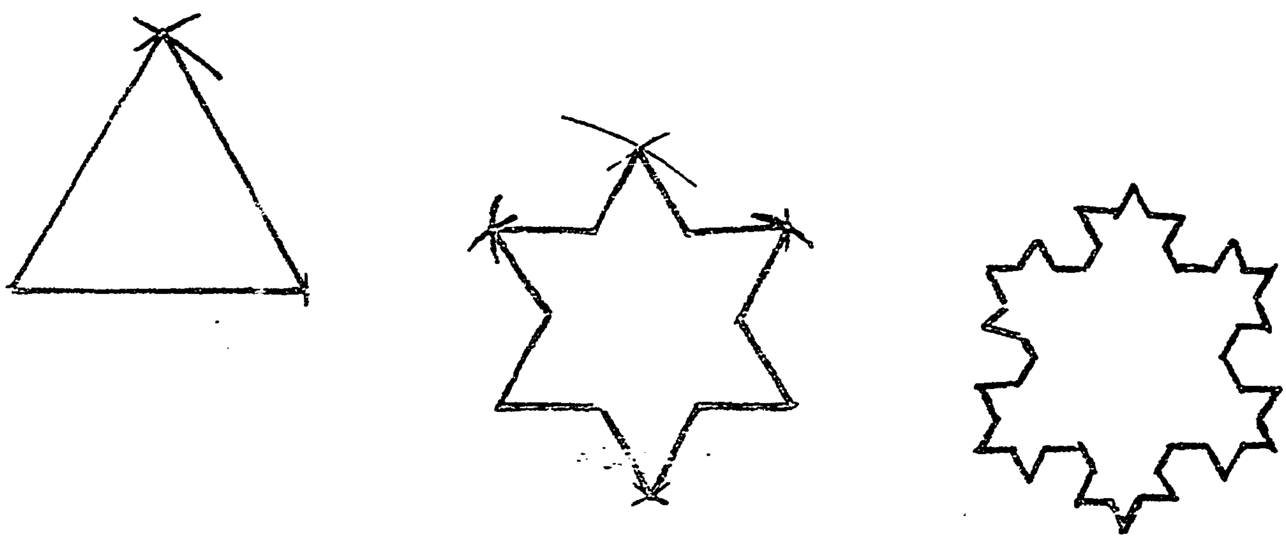
Figure 20. Snowflakes

This is an optional unit which provides pupils with the opportunity to make some unusual constructions. It is a fairly difficult unit and has some interesting questions regarding the area and perimeter of the snowflake shapes. Neatness and accuracy are necessary if the figure is to be attractive.

DRAWING SNOWFLAKES

On one of the large sheets of paper you find in the box, draw an equilateral triangle (all 3 sides equal in length) in the middle of the page. Make each side 9 inches long. Using the compass or the rules in the box, trisect each side of the triangle and on each of the middle thirds erect an equilateral triangle pointing outward. Erase the parts common to the new and old triangles. Trisect each side of the new figure and again upon each middle third erect an equilateral triangle pointing outward. Erase the parts common to the new and old figures. Repeat this process 2 or 3 more times. Now you can take the box of crayons and color the snowflake.

Below are drawings of the first three stages of the snowflake.



Notice the compass marks on drawings 1 and 2.

Do you see an easy way to draw equilateral triangles?

To make another type of snowflake, follow the directions above except draw the small equilateral triangle pointing inward instead of outward.

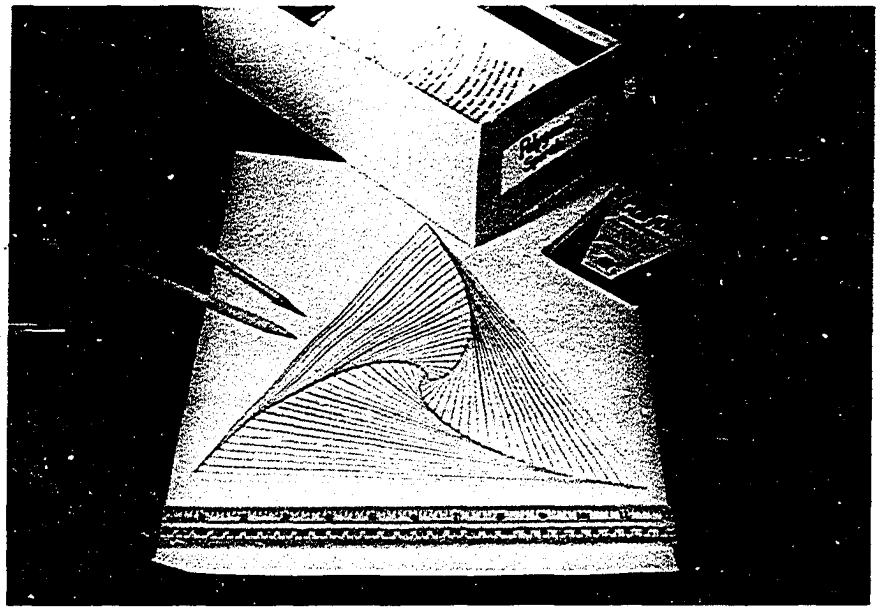


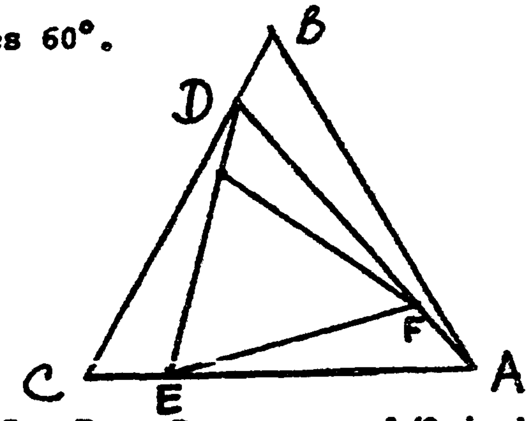
Figure 21. Polygonal spirals

This optional unit demands some precision and neatness if the resulting figure is to look nice. Encourage originality on all construction problems. Squares, hexagons, and the like may be used as a basic shape.

POLYGONAL SPIRALS

To construct a polygonal spiral, first draw an equilateral triangle which is 8 inches on a side. Remember, an equilateral triangle is one with all three sides and all three angles congruent. Each angle measures 60°.

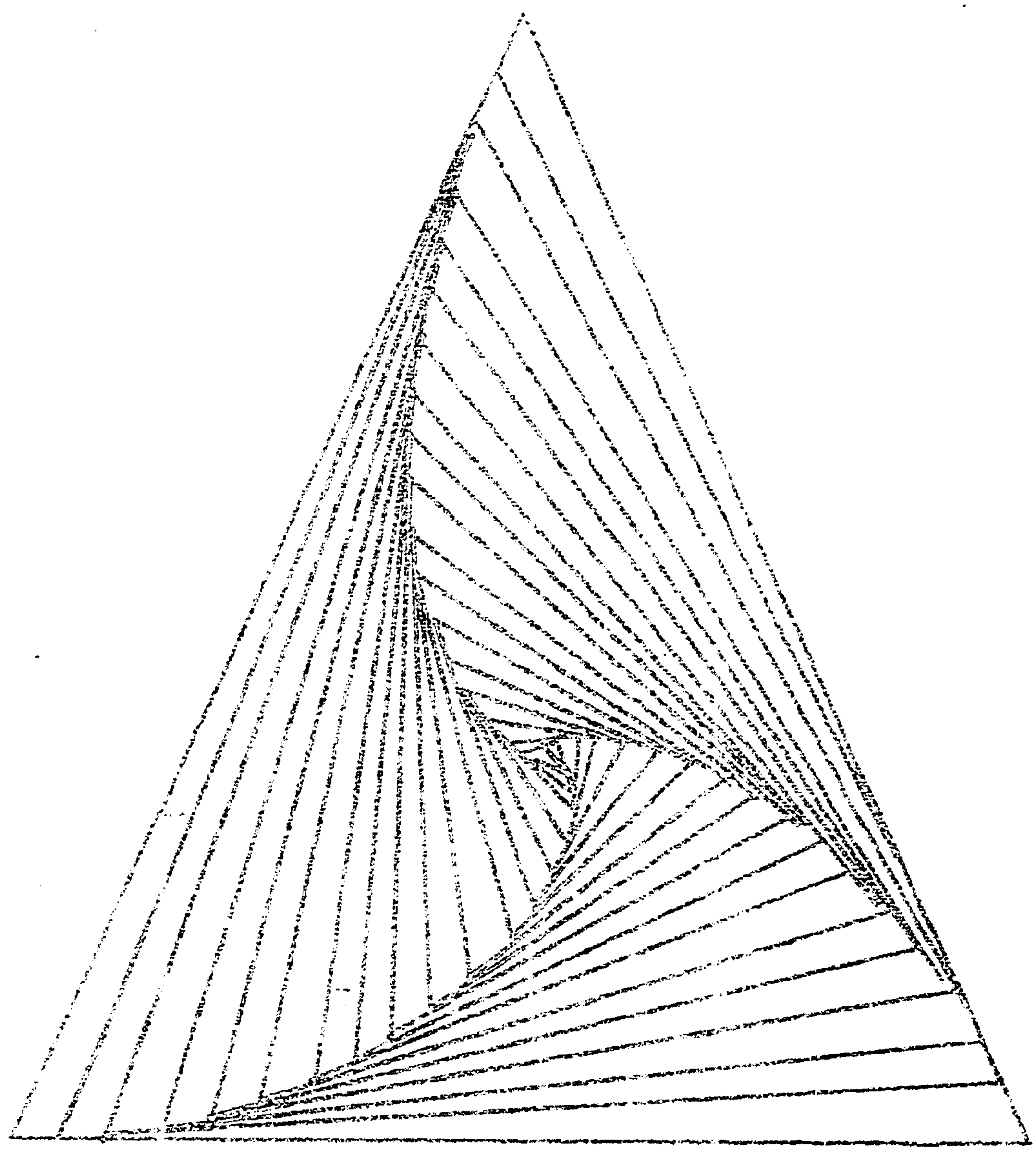
Look at the example.

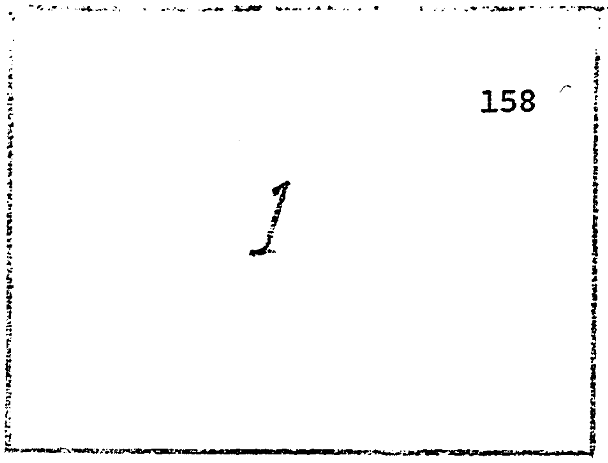


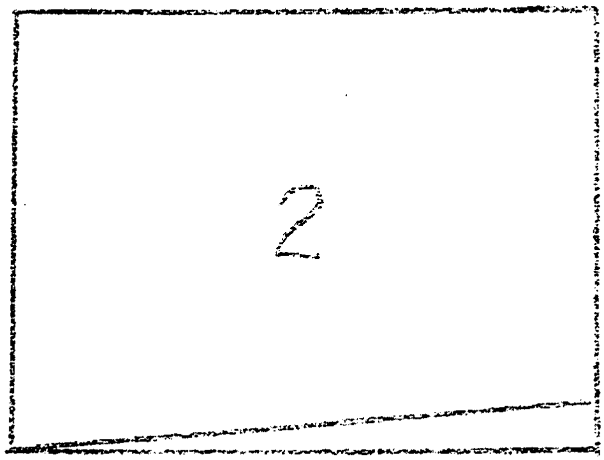
Label your triangle ABC. From B, measure 1/2 inch on $\overline{BC}$ and draw $\overline{AD}$. Now measure 1/2 inch from C on $\overline{CA}$ and draw $\overline{DE}$. Next, measure 1/2 inch from A on $\overline{AD}$ and draw $\overline{EF}$. By continuing this process you will get a polygonal spiral for an equilateral triangle. You can make a better looking spiral by taking points 1/4 inch from the intersections.

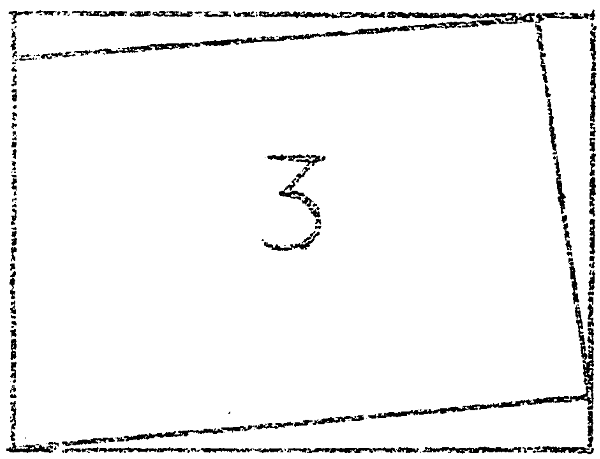
You can also make polygonal spirals for squares, rectangles, hexagons and any figure you choose. In each case you "spiral into" the center by selecting some distance and measuring that distance from each new intersection. The smaller the distance, the more spiraling you will observe. It makes the spiral look better if you cut down on the distance as you get a smaller and smaller shape.

Color the spiral if you wish. We want some of these for the bulletin boards.




1


2


3

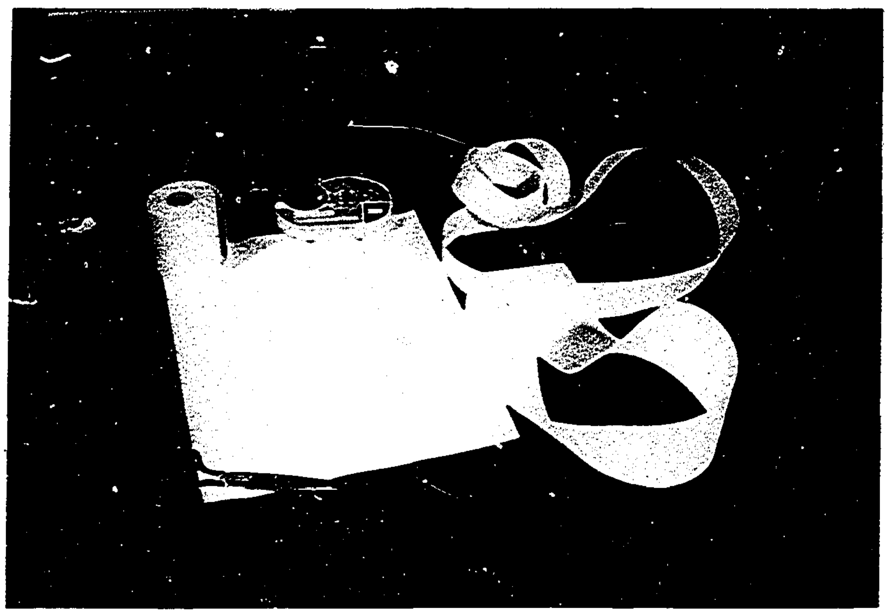


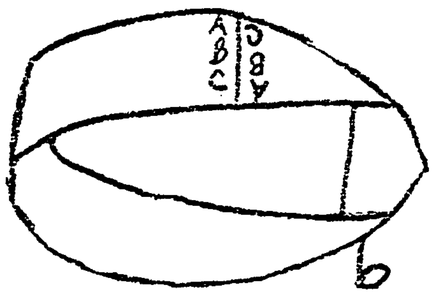
Figure 22. Moebius strip

This optional unit requires that the pupil construct a Moebius strip. Next he will cut strips in a variety of ways and generalize a rule for describing the results.

Have you ever seen a piece of paper with only one surface? There really is such a sheet. It is called a Moebius strip and has been used by many magicians to entertain people. It has been a plaything for mathematicians ever since it was discovered by August Ferdinand Moebius, a German mathematician, in 1858. A fly can walk from any point on this strip to any other point without crossing an edge. Unlike a sheet of paper or a table top, it does not have a top or a bottom, a front or back.

You can make a Moebius strip with any strip of paper. Any size or type of paper will do. We use the strip to make a ring or band, but before we glue the ends together, we give one a half-twist. Attach the band as illustrated.

a



b

If you draw a line on the surface of your Moebius strip, you will find that you will go all around the entire surface without crossing an edge. Paint or color one surface without going over an edge. Is there another surface that remains to be colored?

For another unusual result, cut the band lengthwise along a line in the center of the strip. What unexpected result did you obtain? If you make another band, and cut it lengthwise one-third of the way in from an edge, you will get still a different result.

The Moebius strip enables us to take a new look at right- and left-handed objects like shoes or gloves. If you compare the two gloves of a pair of gloves, you will find that they are equal in all measurements you can make. But you know the gloves are very different. The left-handed glove won't fit your right hand.

How can you change a right-handed glove to a left-handed one? In two dimensions it seems possible on a Moebius strip. If you could slide a picture of the glove along the surface of a Moebius strip, the glove would be upside down and backward when it got back to the starting point.

MOEBIUS STRIP FACTS

You can have fun showing your friends the odd results you get by cutting Moebius strips in different ways. Copy and complete the table below to see what happens when you change the number of twists and the way in which you cut the strip.

Number of Half-twists	Number of Sides and Edges	Kind of cut	Result of Cut (Number of sides and edges, length and width, number of loops, twists and knots)
0		center	
1		center	
1		one-third	
2		center	
2		one-third	
3		center	
3		one-third	

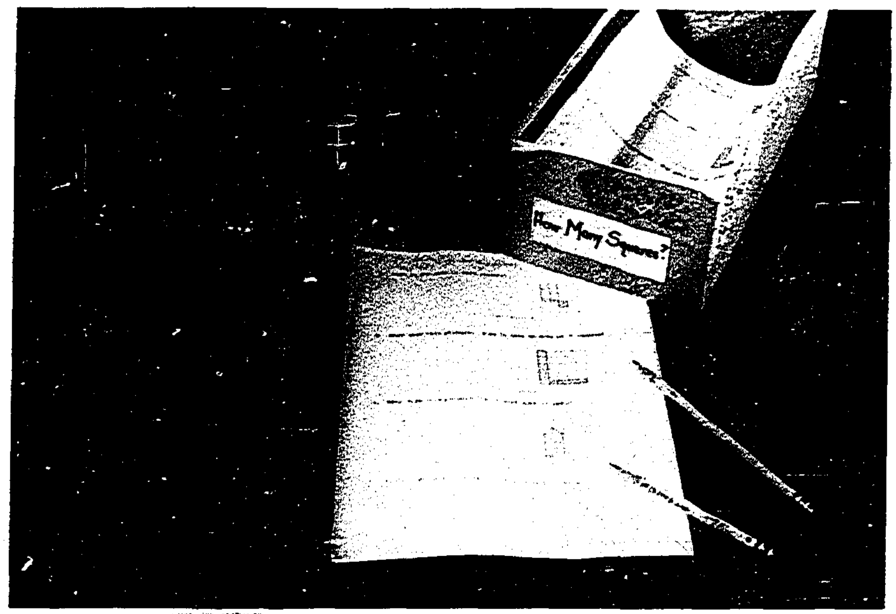
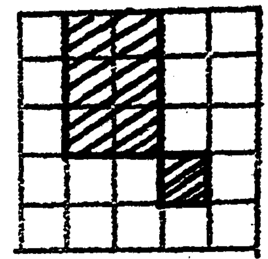


Figure 23. How many squares?

This optional unit is an exercise in counting the number of square units in a given shape. The pupil is also asked to construct some shapes which have a given area.

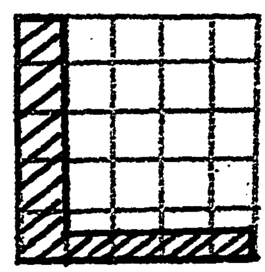
HOW MANY SQUARES?

1. What is the area in this picture?



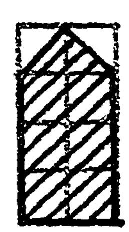
______

2. Marc drew this picture. Does it have an area of 7 squares?



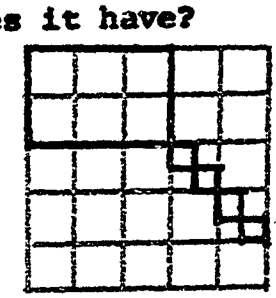
______

3. Mary drew this picture. What area does it have?



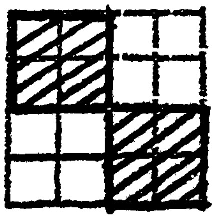
______

4. Jerry drew this picture. What area does it have?



______

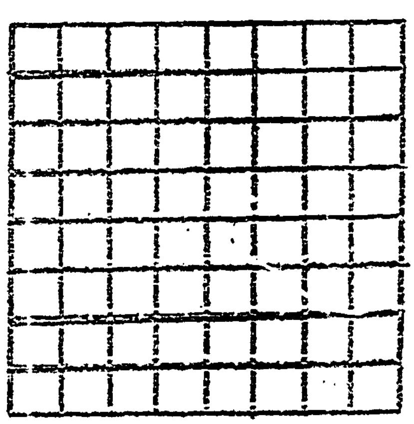
5. **Ellen drew this picture. What area does it have?**



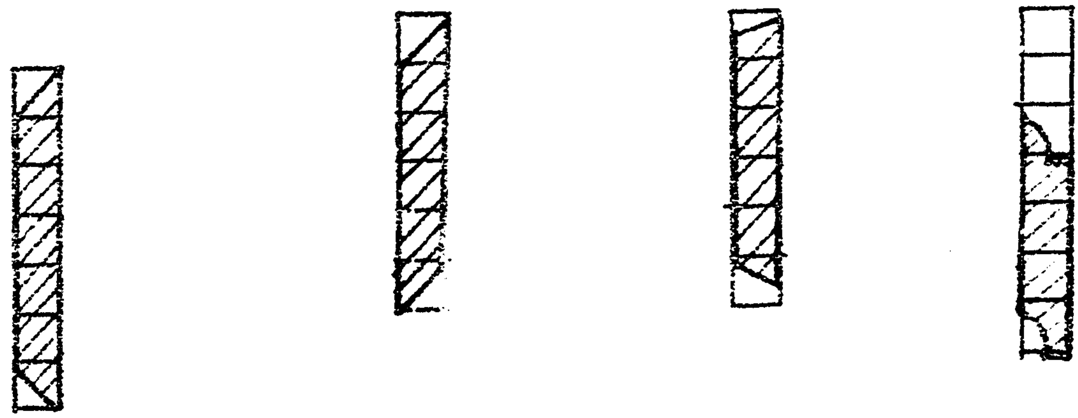
6. **Jim drew this picture. What area does it have?**

7. **Now draw your own picture which has an area of 7 squares.**

Make it different from Marc's and Mary's and Jerry's and Jim's.



8. What is the area of each figure?



________ ________ ________ ________

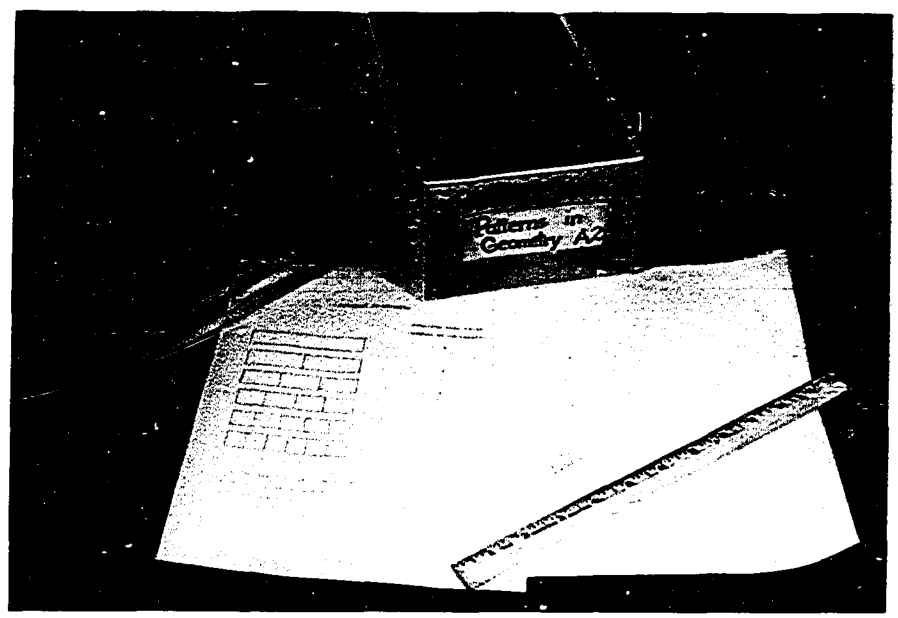
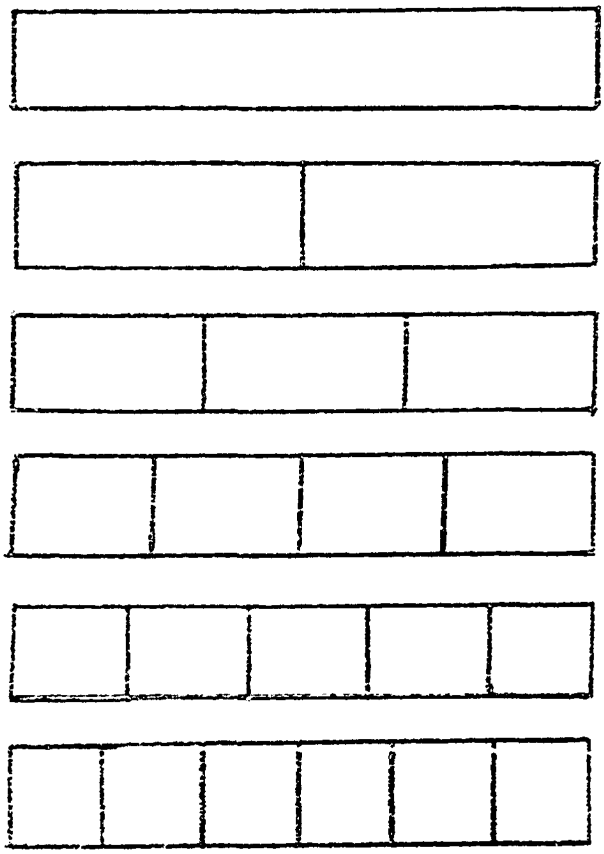


Figure 24. Geometric patterns

This is a series of inductive sequences. Pupils will answer some specific questions and then generalize a pattern.

PATTERNS IN GEOMETRY



Complete this table:

Number of rectangles
1
3
6
______
______
______

Look at your results. Can you see any pattern? Can you find the total number of rectangles for  by using the pattern? How many rectangles? __________ How did you decide?

____

____

Now complete this table.

	Number of points	Number of different line segments
	2	1
A B C	3	3
D E F G	4	______
H I J K L	______	______
M N O P Q R	______	______

How many different line segments when there are 7 points? ______

8 points? ______

9 points? ______

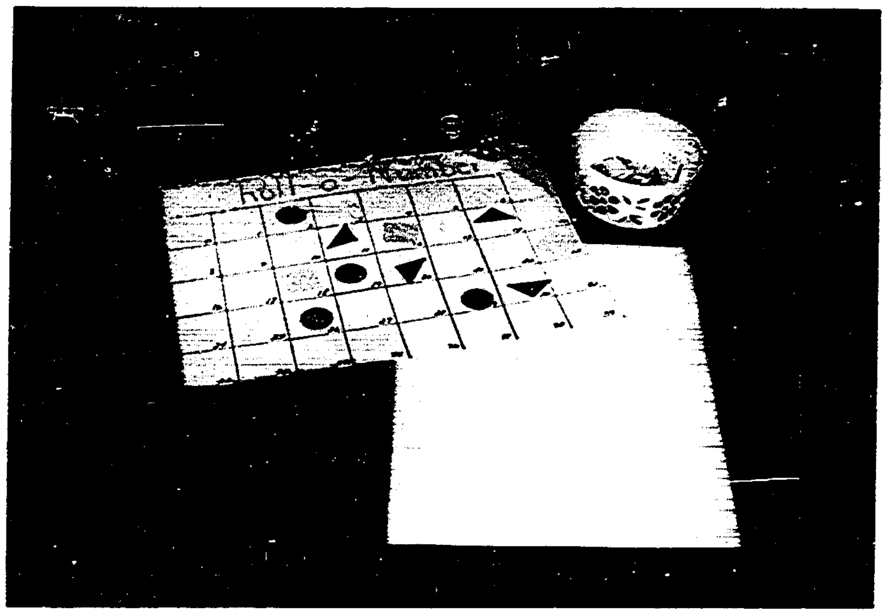


Figure 25. Roll a number

This optional unit is an inductive sequence which is presented in a game format. Two or three pupils can play at one time.

The object of this game is to use the number combination on a roll of the 3 dice. By adding, subtracting, multiplying, and dividing the numbers on the die, name a number on the board. Use each number on the die exactly once.

For example, assume you roll a 3 on the red die, a 2 on the green die and a 4 on the white die. Using these numbers you could name 9, $(3 + 2 + 4) = 9$

or you could name 5, $(4 + 3 - 2) = 5$

or you could name 6, $(4 \times 3) \div 2 = 6$

or you could name 2, $(4 - 2) \div 3 = 2$

or you could name 1, $3 - (4 \div 2) = 1$, and so on.

Place your marker on the number your combination names and then record your points.

You score 1 point for a play on the board. You can score additional points by placing your marker adjacent to other markers. You get one additional point for each adjacent marker.

If you cannot name an open number on the board, you must pass. Three passes and you are finished with the game. When each player has passed 3 times the game is over. The player with the most points is the winner.

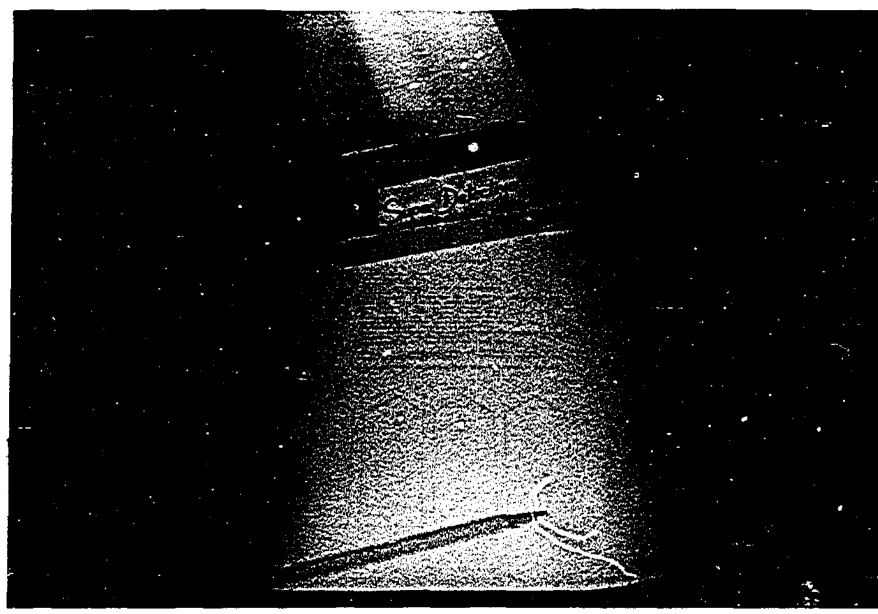


Figure 26. Super detective

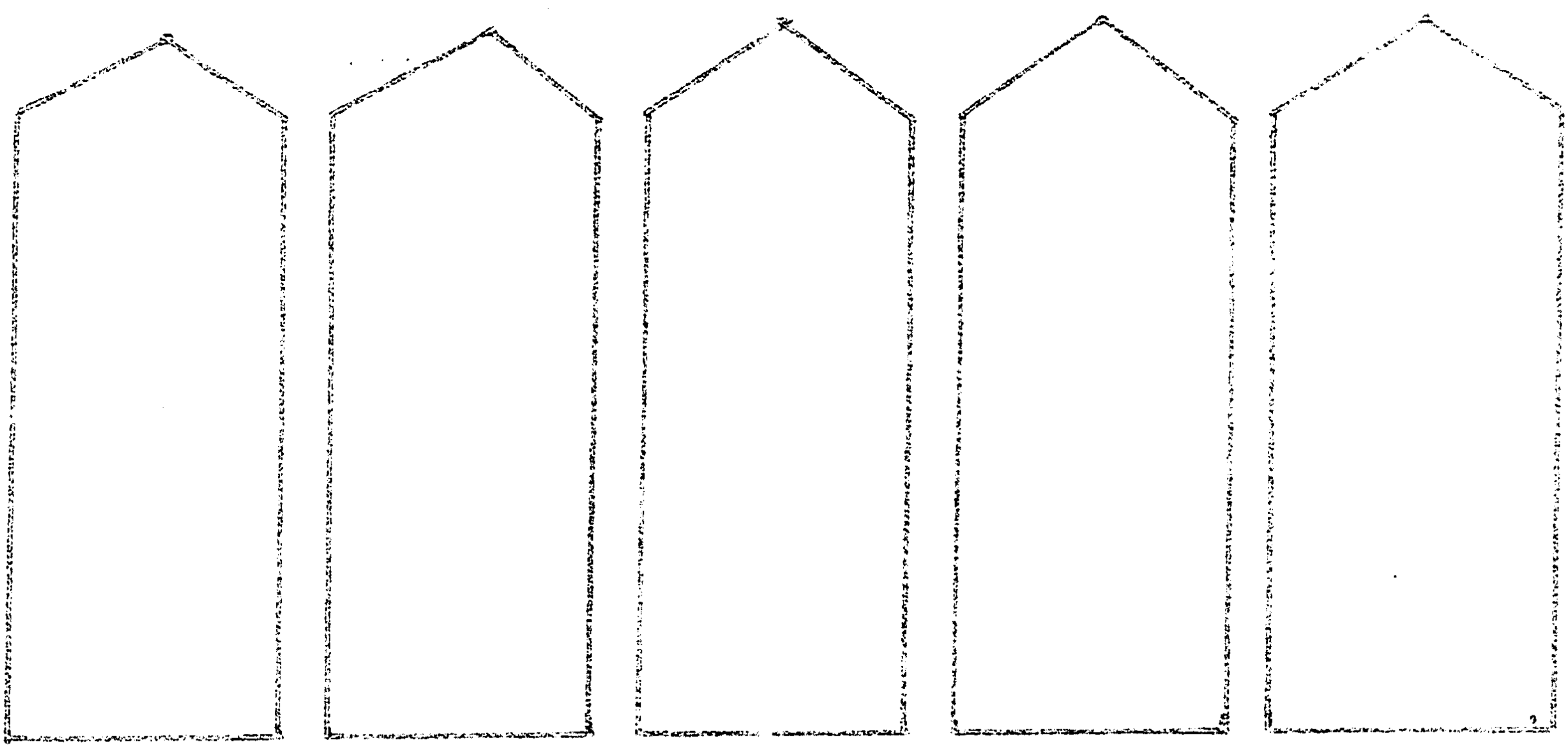
This optional unit is an exercise in logic and determination. It is important that pupils keep a systematic account of the known data. Pupils may need encouragement to keep them searching for the solution.

SUPER DETECTIVE

Five houses in a row.
Englishman lives in the red house.
Spaniard owns a dog.
Coffee is drunk in the green house.
Ukranian drinks tea.
Green house is just to the right of the Ivory house.
Man who drives Buick owns snails.
Ford owner lives in the yellow house.
Man in the middle house drinks milk.
Norwegian lives in the first house.
Chevy owner lives next to the man with a fox.
Plymouth owner drinks orange juice.
Ford owner lives next to the man with a horse.
Japanese owns an Oldsmobile.
Norwegian lives next to the blue house.

There are 5 nationalities, 5 different pets, 5 different cars, 5 different drinks, 5 different colored houses.

Problem: WHO DRINKS WATER AND WHO OWNS A ZEBRA?



APPENDIX B

Mathematics Laboratory Assignments

Shoe box	Team 1	Team 2	Team 3	Team 4	Team 5	Team 6	Team 7	Team 8	Team 9
Angle measurement	T/24	Th/19	W/18	T/17	M/16	F/13	Th/12	W/11	T/10
Square puzzle	W/25	T/24	Th/19	W/18	T/17	M/16	F/13	Th/12	W/11
Stellar polygons	Th/26	W/25	T/24	Th/19	W/18	T/17	M/16	F/13	Th/12
Curve stitching	F/27	Th/26	W/25	T/24	Th/19	W/18	T/17	M/16	F/13
Mirror geometry	M/2	F/27	Th/26	W/25	T/24	Th/19	W/18	T/17	M/16
Tower of Hanoi	T/3	M/2	F/27	Th/26	W/25	T/24	Th/19	W/18	T/17
Calculation of pi (π)	W/4	T/3	M/2	F/27	Th/26	W/25	T/24	Th/19	W/18
Volume relationship	Th/5	W/4	T/3	M/2	F/27	Th/26	W/25	T/24	Th/19
Area of a rectangle	F/6	Th/5	W/4	T/3	M/2	F/27	Th/26	W/25	T/24
Area of a right triangle	M/9	F/6	Th/5	W/4	T/3	M/2	F/27	Th/26	W/25
Area of a parallelogram	T/10	M/9	F/6	Th/5	W/4	T/3	M/2	F/27	Th/26
Area of a triangle	W/11	T/10	M/9	F/6	Th/5	W/4	T/3	M/2	F/27
Area and perimeter	Th/12	W/11	T/10	M/9	F/6	Th/5	W/4	T/3	M/2
Side-area relationships	F/13	Th/12	W/11	T/10	M/9	F/6	Th/5	W/4	T/3
Rectangular prisms	M/16	F/13	Th/12	W/11	T/10	M/9	F/6	Th/5	W/4
Surface area	T/17	M/16	F/13	Th/12	W/11	T/10	M/9	F/6	Th/5
Construction of polyhedra	W/18	T/17	M/16	F/13	Th/12	W/11	T/10	M/9	F/6
Euler's formula	Th/19	W/18	T/17	M/16	F/13	Th/12	W/11	T/10	M/9